Review of Selected Topics in Probability
Conditional Probability

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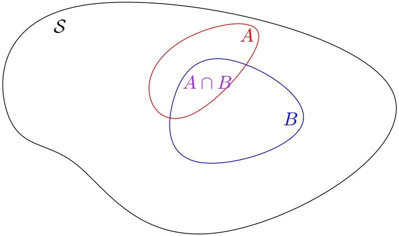
Lecture 3

Conditional Probability

- Event of interest: A
- Additional information (*after the realization of the event but before total disclosure* - "early" event): B, such that $\Pr(B) > 0$.



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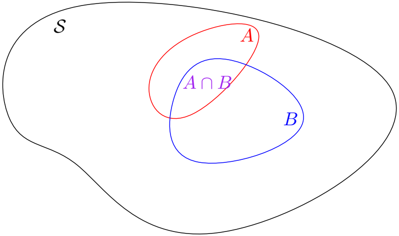

The probability of A given B is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{1}$$

Some Examples

Example 1: Assume we roll a symmetric 6-sided die. Let A be the event that we roll $3, 4$ or 6 and let B be the event that we roll an even number. Then $\Pr(A) = \frac{1}{2}$, $\Pr(A|B) = \frac{2}{3}$ and $\Pr(A|\bar{B}) =$

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Example 2: Assume someone chooses randomly a person from a class of 8 women (6 have a degree from U. Patras and there is 1 biologist) and 3 men (1 has a degree from U. Patras). Consider the following events:

- A: The person chosen has a degree from U. Patras.
- B: The person chosen is a woman.
- C: The person chosen is a biologist.

Then $\Pr(A) = \frac{7}{11}$, $\Pr(A|B) = \frac{6}{8}$, $\Pr(\bar{A}|B) = \frac{2}{8}$,
$\Pr(A|\bar{B} \cap C) = \text{undefined}$, $\Pr(B|A) = \frac{6}{7}$, $\Pr(B|\bar{A}) = \frac{2}{4}$.

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What happens if, by some error, a person appears more than once in our class list? What if a person is more likely to get picked?

Conditional Probability - Properties

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Theorem (Product Rule)

For events $A_1, \ldots, A_n$,

$$\Pr(A_1 A_2 \cdots A_n) = \Pr(A_1)\Pr(A_2|A_1)\cdots\Pr(A_n|A_1\cdots A_{n-1}) \quad (2)$$

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Theorem (Law of Total Probability)

Let $B_1, \ldots, B_n$ be a partition of $\mathcal{S}$, then

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A|B_i)\Pr(B_i). \quad (3)$$

a posteriori Probability

- Example 1: Suppose we have two bins C_1 and C_2. The first one has 2 blue balls and 1 red and the second one has 1 blue ball and 3 red ones. We pick one of the two bins equiprobably and we choose a random ball from it. Given that it is red, what is the probability that it came from the first bin?

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- Example 2 - diagnostic tests: Let B be the event that a person in a population has some disease and let A be the event that a specific testing procedure for this disease becomes positive.
  - We are interested in $\Pr(B|A)$.
  - Sensitivity and Specificity: We usually know $\Pr(A|B)$ (true positive) and $1 - \Pr(A|\bar{B})$ (true negative).
  - We also assume we know $\Pr(B)$ and $\Pr(\bar{B})$.

Bayes Theorem - *a posteriori* Probability

- We are interested in the probability of an event B, given some "later" event A, and we know the probability $\Pr(A|B)$ and $\Pr(A|\bar{B})$; same mechanic, only conceptually different. Then

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(B)\Pr(A|B)}{\Pr(B)\Pr(A|B) + \Pr(\bar{B})\Pr(A|\bar{B})}$$

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Theorem (Bayes Theorem)

Let $B_1, \ldots, B_n$ be a partition of the sample space and assume $\Pr(A) > 0$ and $\Pr(B_i) > 0, \forall i$. Then

$$\Pr(B_i|A) = \frac{\Pr(B_i)\Pr(A|B_i)}{\sum_{j=1}^{n}\Pr(B_j)\Pr(A|B_j)}. \quad (4)$$

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Suppose we have two bins C_1 and C_2. The first one has 2 blue balls and 1 red and the second one has 1 blue ball and 3 red ones. We pick one of the two bins (i) equiprobably and (ii) with probability proportional to the number of balls it has. We then choose a random ball from it. Given that it is red, what is the probability that it came from the first bin?

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(a) $\frac{1}{7}$ and $\frac{4}{13}$.

(b) $\frac{4}{13}$ and $\frac{1}{4}$

(c) The second probability is smaller than the first.

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Answer: (b)

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An event A is independent of an event B iff

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Easier Product Rule: If A, B are independent, then $\Pr(AB) = \Pr(A)\Pr(B)$. (Extremely useful - see e.g. balls and bins quiz of previous lecture - but be careful when assuming independence!)

Independence (cntd.)

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Events $A_1, \ldots, A_n$ are mutually independent iff, for any $k \in [n]$,

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Then $\Pr(A_i A_j) = \Pr(A_i) \Pr(A_j) = \frac{1}{9}$, for all $i \neq j$, but
$\Pr(A_1 A_2 A_3) = \frac{1}{9} \neq \Pr(A_1) \Pr(A_2) \Pr(A_3)$.

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- Testing for mutual independence is hard (even experts in probability cannot tell sometimes); we have to check whether

$$\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - n - 1$$

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- We usually assume mutual independence when events *happen in different time and space*.

The Birthday Paradox

Assuming all birthdays are equiprobable and that they are mutually independent, what is the probability that at least 2 individuals in a class of $m = 23$ people are born on the same day of the year (assume a year has $N = 365$ days)?

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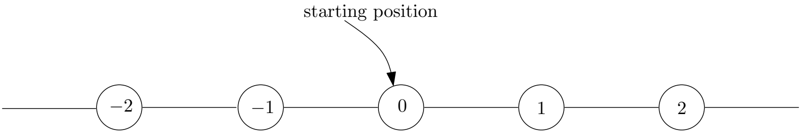
Answer: Homework! (You can reduce this to a balls and bins problem; Use conditional probability)

Random Walk on the Line



An individual is placed at vertex 0. At each time step $t = 1, 2, \ldots$, he independently decides to move either one vertex to its right or one vertex to its left. What is the probability that after n steps he is back where he started?

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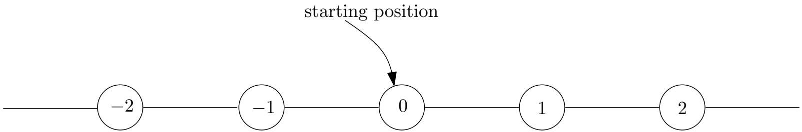

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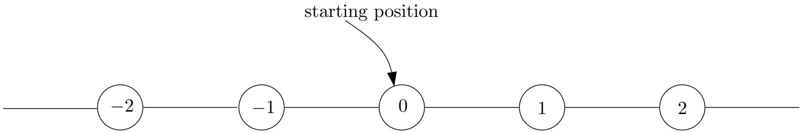

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Further reading

S. Ross. A first course in probability:
Chapter 3, "Conditional Probability and Independence"