PRIMER FOR MANIFOLD THEORY

After imposing rectilinear coordinates on a Euclidean space E^n (such as the plane E^2) we identify Euclidean space with $\mathbb{R}^n$, the vector space of $n-$tuples of numbers. In fact, since a Euclidean space, in this sense, is an object of intuition (at least in 2d and 3d) some readers may insist that to be sure such a space of points really exists, we should start with $\mathbb{R}^n$ and then "forget" the origin and all the vector space structure while retaining the notion of point and distance. The coordinatization of Euclidean space is then just a "remembering" of this forgotten structure. Thus our coordinates arise from a map $x : E^n \to \mathbb{R}^n$ which is just the identity map. This approach has much to recommend it but there is at least one regrettable aspect to this approach and that is the psychological effect that occurs when we impose *other* coordinates on our space and then introduce differentiable manifolds as abstract geometric objects that support coordinate systems. It might then seem that this is a big and new abstraction. When the definitions of charts and atlases and so on appear, a certain notational fastidiousness sets in that somehow creates a psychological gap between open sets in $\mathbb{R}^n$ and the abstract spaces that we coordinatize. But what is now lost from sight is that we have already been dealing with an abstract manifold! Namely, E^3 which support many coordinate systems such as spherical coordinates. **Putting coordinates on space, even the rectangular coordinates which allows us to identify E^3 with R^3 is already the basic idea involved in the notion of a differentiable manifold.** The idea of a differentiable manifold is a natural idea that becomes overly complicated when we are forced to make exact definitions. As a result of the nature of these definition the student is faced with a pedagogy that teaches notation and trains one to examine each expression for logical set theoretic self consistency, but fails to teach geometric intuition. Having made this complaint, the author must confess that he too will use the modern notation and will not stray far from standard practice. These remarks are meant to encourage the student to stop and seek the simplest most intuitive viewpoint whenever feeling overwhelmed by notation. The student is encouraged to experiment with abbreviated personal notation when checking calculations and to draw diagrams and schematics that encode the geometric ideas whenever possible. "The picture writes the equations".

So, as we said, after imposing rectilinear coordinates on a Euclidean space E^n (such as the plane E^2 space E^3) we identify Euclidean space with $\mathbb{R}^n$, the vector space of $n-$tuples of numbers.

We will envision there to be a copy of $\mathbb{R}^n$ at each of its points $p \in \mathbb{R}^n$ which is denoted $\mathbb{R}^n_p$. The elements of $\mathbb{R}^n_p$ are to be thought of as vectors based at p, that is, the "tangent vectors" at p. These tangent spaces are related to each other by the obvious notion of vectors being parallel (this is exactly what is not generally possible for tangents spaces of a general manifold). For the standard basis vectors e_j (relative to the coordinates x_i) taken as being based at p we often write $\left.\frac{\partial}{\partial x_i}\right|_p$ and this has the convenient second interpretation as a differential operator acting on smooth functions defined near $p \in \mathbb{R}^n$. Namely,

$$\left.\frac{\partial}{\partial x_i}\right|_p f = \frac{\partial f}{\partial x_i}(p).$$

An n-tuple of smooth functions $X^1, ..., X^n$ defines a smooth vector field $X = \sum X^i \frac{\partial}{\partial x_i}$ whose value at p is $\sum X^i(p) \left.\frac{\partial}{\partial x_i}\right|_p$. Thus a vector field assigns to each p in its domain, an open set U, a vector $\sum X^i(p) \left.\frac{\partial}{\partial x_i}\right|_p$ at p. We may also think of vector field as a differential operator via

$$f \mapsto Xf \in C^\infty(U)$$
$$(Xf)(p) := \sum X^i(p) \frac{\partial f}{\partial x_i}(p)$$

Example 1 $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ *is a vector field defined on* $U = \mathbb{R}^2 - \{0\}$ *and* $(Xf)(x, y) = y\frac{\partial f}{\partial x}(x, y) - x\frac{\partial f}{\partial y}(x, y)$.

Notice that we may certainly add vector fields defined over the same open set as well as multiply by functions defined there:

$$(fX + gY)(p) = f(p)X(p) + g(p)X(p)$$

The familiar expression $df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n$ has the intuitive interpretation expressing how small changes in the variables of a function give rise to small changes in the value of the function. Two questions should come to mind. First, "what does 'small' mean and how small is small enough?" Second, "which direction are we moving in the coordinate" space? The answer to these questions lead to the more sophisticated interpretation of *df* as being a linear functional on each tangent space. Thus we must choose a direction v_p at $p \in \mathbb{R}^n$ and then $df(v_p)$ is a number depending linearly on our choice of vector v_p. The definition is determined by $dx_i(e_j) = \delta_{ij}$. In fact, this shall be the basis of our definition of *df* at p. We want

$$Df|_p \left(\left.\frac{\partial}{\partial x_i}\right|_p \right) := \frac{\partial f}{\partial x_i}(p).$$

Now any vector at p may be written $v_p = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x_i}\right|_p$ which invites us to use v_p as a differential operator (at p):

$$v_p f := \sum_{i=1}^n v^i \frac{\partial f}{\partial x_i}(p) \in \mathbb{R}$$

This consistent with our previous statement about a vector field being a differential operator simply because $X(p) = X_p$ is a vector at p for every $p \in U$. This is just the directional derivative. In fact we also see that

$$Df|_p(v_p) = \sum_j \frac{\partial f}{\partial x_j}(p)dx_j \left(\sum_{i=1}^n v^i \left.\frac{\partial}{\partial x_i}\right|_p \right)$$
$$= \sum_{i=1}^n v^i \frac{\partial f}{\partial x_i}(p) = v_p f$$

so that our choices lead to the following definition:

Definition 2 *Let f be a smooth function on an open subset U of $\mathbb{R}^n$. By the symbol df we mean a family of maps $Df|_p$ with p varying over the domain U of f and where each such map is a linear functional of tangent vectors based at p given by $Df|_p(v_p) = v_p f = \sum_{i=1}^n v^i \frac{\partial f}{\partial x_i}(p)$.*

Definition 3 *More generally, a smooth 1-form α on U is a family of linear functionals $\alpha_p : T_p\mathbb{R}^n \to \mathbb{R}$ with $p \in U$ that is smooth is the sense that $\alpha_p(\left.\frac{\partial}{\partial x_i}\right|_p)$ is a smooth function of p for all i.*

¿From this last definition it follows that if $X = X^i \frac{\partial}{\partial x_i}$ is a smooth vector field then $\alpha(X)(p) := \alpha_p(X_p)$ defines a smooth function of p. Thus an alternative way to view a $1-$form is as a map $\alpha : X \mapsto \alpha(X)$ that is defined on vector fields and linear over the algebra of smooth functions $C^\infty(U)$:

$$\alpha(fX + gY) = f\alpha(X) + g\alpha(Y).$$

0.1 Fixing a problem

Now it is at this point that we want to destroy the privilege of the rectangular coordinates and express our objects in an arbitrary coordinate system smoothly related to the existing coordinates. This means that for any two such coordinate systems, say $u^1, ..., u^n$ and $y^1, ..., y^n$ we want to have the ability to express fields and forms in either system and have for instance

$$X^i_{(y)} \frac{\partial}{\partial y_i} = X = X^i_{(u)} \frac{\partial}{\partial u_i}$$

for appropriate functions $X^i_{(y)}$, $X^i_{(u)}$. This equation only makes sense on the overlap of the domains of the coordinate systems. To be consistent with the chain rule we must have

$$\frac{\partial}{\partial y^i} = \frac{\partial u^j}{\partial y^i} \frac{\partial}{\partial u^j}$$

which then forces the familiar transformation law:

$$\sum \frac{\partial u^j}{\partial y^i} X^i_{(y)} = X^i_{(u)}$$

We think of $X^i_{(y)}$ and $X^i_{(u)}$ as referring to or representing the same geometric reality from two different coordinate systems. No big deal right? OK, how about the fact that there is this underlying abstract space that we are coordinatizing? That too is no big deal. We were always doing it in calculus anyway. What about the fact that the coordinate systems aren't defined as a 1-1 correspondence with the points of the space unless we leave out some points in some coordinates. For example, when using polar coordinates we leave out the origin and the axis we are measuring from to avoid ambiguity in θ and in order to have a nice open domain. Well if this is all fine then we may as well imagine other abstract spaces that support coordinates in this way. In fact, we don't have to look far for an example. Any surface such as the sphere will do. We can talk about 1-forms like say $\alpha = \theta d\phi + \phi \sin(\theta) d\theta$, or a vector field tangent to the sphere $\theta \sin(\phi) \frac{\partial}{\partial \theta} + \theta^2 \frac{\partial}{\partial \phi}$ and so on (just pulling things out of a hat). We just have to be clear about how these arise and most of all how to change to a new coordinate expression for the same object in a different coordinate system. This is the approach of tensor analysis. An object called a 2-tensor T is represented in two different coordinate systems as for instance

$$T = \sum T^{ij}_{(y)} \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial y^j}$$
$$T = \sum T^{ij}_{(u)} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}$$

where all we really need to know for many purposes is the transformation law for coordinate changes:

$$T^{ij}_{(y)} = \sum_{r,s} T^{rs}_{(u)} \frac{\partial y^i}{\partial u^r} \frac{\partial y^i}{\partial u^s}.$$

Then either expression is referring to the same abstract tensor T. This is just a preview but it highlights the approach wherein a transformation laws play a defining role. Eventually, this leads to the abstract notion of a G-bundle.