IMC preparation seminar, Day 1

November 6, 2023

1 Theory/Background

Invariants: Invariants(semi-invariants) are quantities that remain constant(decrease or increase) under some transformation and characterize a process, a set etc. Both invariants and semi-invariants may hide behind problems with repeatedly applied transformations or algorithms.

Dirichlet's box principle: If $kn+1$ objects(pigeons) are distributed among n sets(holes), one of the sets will contain at least $k+1$ objects.

Extremal 'principle': An elegant, but hard to apply problem solving method is finding an element of some set that maximizes some property-usually by construction- and using its maximality to prove inequalities or disprove assertions.

2 Problems

2.1 warm up problems

Problem 1 Given 50 distinct positive integers strictly less than 99, prove that some two of them sum to 99.

Problem 2 Consider five 1's and four 0's arranged around a circle. In each step we write the number 0 between two consecutive slots if their numbers are equal and 1 otherwise. Then we erase the previous numbers ending up with nine 0's and 1's in total. Is it possible to end up with nine 0's in finitely many steps? Generalize...

Problem 3 Consider the integers from 1 to 10^5. In each step we replace all numbers with the sum of their digits until we end up with 10^5 one-digit numbers. Will there be more 1's or 2's?

Problem 4 Let $x_1, x_2, x_3, \ldots$ be a sequence of integers such that $1 = x_1 < x_2 < x_3 < \ldots$ and $x_{n+1} \leq 2n$ for $n = 1, 2, 3, \ldots$.. Show that every positive integer k is equal to $x_i - x_j$ for some i and j .

Problem 5 Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

2.2 harder problems

Problem 6 Consider n non-collinear points on the plane. Prove that there exists a straight line passing through exactly 2 of them.

Problem 7 Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.

Problem 8 Consider n distinct positive integers $x_1, x_2, .., x_n$ and the transformation T such that

$$T(x_1, x_2, .., x_n) = \left(\frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, .., \frac{x_n + x_1}{2} \right).$$

Prove that after repeatedly applying T we will end up with at least one non integral component.

Problem 9 In some country all roads between cities are one-way and such that once you leave a city there is no turning back. Prove that there exists a city into which all roads enter and a city from which all roads exit.

Problem 10 Each point of the plane with integer coordinates is assigned an integer, such that each of these numbers is the arithmetic mean of the numbers of the four neighbouring points. Prove that all these integers are equal.

Problem 11 A lion and a Christian in a closed circular Roman arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal? In other words, can the lion catch the Christian in finite time?