1.6.1

Tuesday, June 30, 2026 3:08 PM

$a_1 \ge a_2 \ge \dots \ge a_n \ge \dots \ge 0$ $\quad s_n = \sum_{k=1}^{n} \varepsilon_k a_k$: convergent, $|\varepsilon_k| = 1$. Show $\lim_{n\to\infty} (\varepsilon_1 + \dots + \varepsilon_n) a_n = 0$

$$\sum_{k=m}^{n} \varepsilon_k = \sum_{k=m}^{n} \frac{\varepsilon_k a_k}{a_k} = \sum_{k=m}^{n} \frac{s_k - s_{k-1}}{a_k} = \sum_{k=m}^{n} \frac{s_k}{a_k} - \sum_{k=m}^{n} \frac{s_{k-1}}{a_k} = \sum_{k=m}^{n} \frac{s_k}{a_k} - \sum_{k=m-1}^{n-1} \frac{s_k}{a_{k+1}} = \frac{s_n}{a_n} - \frac{s_{m-1}}{a_m} + \sum_{k=m}^{n-1} \left(\frac{s_k}{a_k} - \frac{s_k}{a_{k+1}} \right) =$$

$$\frac{s_n}{a_n} - \frac{s_{m-1}}{a_m} + \sum_{k=m}^{n-1} \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) s_k = \cancel{\frac{s_n}{a_n}} - \frac{s_{m-1}}{a_m} + \sum_{k=m}^{n-1} \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) (s_k - s_n) + \sum_{k=m}^{n-1} \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) s_n$$

$$\to \left(\frac{1}{a_m} - \cancel{\frac{1}{a_n}} \right) s_n$$

$\varepsilon > 0$: $\exists N_0 \in \mathbb{N}$: $|s_N - s_M| < \varepsilon \quad \forall N, M > N_0$

$m > N_0$

$|s_k - s_n|$

$$\left| \sum_{k=m}^{n} \varepsilon_k \right| \le \left| \frac{s_{m-1}}{a_m} \right| + \sum_{k=m}^{n-1} \left| \frac{1}{a_k} - \frac{1}{a_{k+1}} \right| \varepsilon + \left| \frac{s_n}{a_m} \right| = \left| \frac{s_{m-1}}{a_m} \right| + \left| \frac{s_n}{a_m} \right| + \varepsilon \left(\frac{1}{a_n} - \frac{1}{a_m} \right)$$

$$a_n \left| \sum_{k=1}^{n} \varepsilon_k \right| \le a_n \left| \sum_{k=1}^{m-1} \varepsilon_k \right| + a_n \left| \sum_{k=m}^{n} \varepsilon_k \right| \le \underbrace{a_n (m-1)}_{\to 0} + \underbrace{a_n \left(\left| \frac{s_{m-1}}{a_m} \right| + \left| \frac{s_n}{a_m} \right| \right)}_{\to 0} + \underbrace{\varepsilon \left(1 - \frac{a_n}{a_m} \right)}_{\to \varepsilon}$$

$$\limsup_{n\to\infty} a_n \left| \sum_{k=1}^{n} \varepsilon_k \right| \le \varepsilon \quad \forall \varepsilon > 0$$

$$\text{VI}$$

$$\liminf a_n \left| \sum_{k=1}^{n} \varepsilon_k \right| \ge 0$$

1.6.2

Tuesday, June 30, 2026 3:46 PM

$a_m + a_n - 1 < a_{m+n} < a_m + a_n + 1$

$2a_n - 1 < a_{2n} < 2a_n + 1$

$a_{2^k} < 2a_{2^{k-1}} + 1 < 2(2a_{2^{k-2}} + 1) + 1 < \dots < 2^k a_1 + (1 + 2 + \dots + 2^{k-1})$

$2^k a_1 - 2^k + 1 < a_{2^k} < 2^k a_1 + 2^k - 1 \Rightarrow a_1 - 1 + \frac{1}{2^k} < \frac{a_{2^k}}{2^k} < a_1 + 1 - \frac{1}{2^k}$

$\left| \frac{a_{2^{k+1}}}{2^{k+1}} - \frac{a_{2^k}}{2^k} \right| < \frac{1}{2^{k+1}}$

$\left(\frac{a_{2^n}}{2^n} \right)$

$|a_n - \ell| \leq |a_n - a_{k_m}| + |a_{k_m} - \ell| < \varepsilon$

$\wedge$ $\frac{\varepsilon}{2}$ $\qquad$ $\wedge$ $\frac{\varepsilon}{2}$

$n, k_m > N_1$ $\qquad$ $k_m > N_2$

$\vee$

m

$n, k_m > N_0 = \max(N_1, N_2)$

$\left| \frac{a_{2^n}}{2^n} - \frac{a_{2^m}}{2^m} \right| = \left| \frac{a_{2^n}}{2^n} - \frac{a_{2^{n-1}}}{2^{n-1}} + \dots + \frac{a_{2^{m+1}}}{2^{m+1}} - \frac{a_{2^m}}{2^m} \right| \leq \frac{1}{2^n} + \dots + \frac{1}{2^{m+1}} \leq \frac{1}{2^{m+1}} + \dots + \frac{1}{2^n} + \dots = \frac{1}{2^m} \to 0$

$a_n = a_{b_k 2^k + \dots + b_1 2 + b_0} < a_{b_k 2^k} + \dots + a_{b_1 2} + a_{b_0} + k$

$\sum_{i=0}^{k} b_i a_{2^i} \cdot + \left(\sum_{i=0}^{k} b_i - 1 \right) < \sum_{i=0}^{k} \left(b_i 2^i a_1 + 2^i - 1 \right) + \left(\sum b_i - 1 \right) = na_1 + 2^{k+1} - 1 - k - 1 + \left(\sum b_i - 1 \right)$

$\frac{a_{mk+r}}{mk+r} - \frac{a_m}{m} < \frac{a_{mk} + a_r + 1}{mk+r} - \frac{a_m}{m} < \frac{ka_m + a_r + k}{mk+r} - \frac{a_m}{m} = \frac{mka_m + ma_r + mk - (mk+r)a_m}{m(mk+r)} = \frac{ma_r - ra_m + mk}{m(mk+r)} = \frac{r}{mk+r}\left(\frac{a_r}{r} - \frac{a_m}{m} \right) + \frac{k}{mk+r} < \frac{k}{mk+r}\left(\frac{a_r}{r} - \frac{a_m}{m} \right) + \frac{1}{m} \leq \frac{1}{m}\left(\frac{a_r}{r} - \frac{a_m}{m} \right) + \frac{1}{m} \to 0$

$\wedge$

$\frac{k}{mk+r}$

$a_1 - \frac{2^{k+1} - 2}{n} < \frac{a_n}{n} < a_1 + \frac{2^{k+1} - 2}{n}$

$\wedge$ $\qquad$ $\wedge$

$a_1 - 2$ $\qquad$ $a_1 + 2$

$2^k < n < 2^{k+1} \Rightarrow 1 < \frac{2^{k+1}}{n} < 2$

$m = kn + r$

$na_1 - n + 1 < a_n < a_{n-1} + a_1 + 1 < \dots < na_1 + n - 1$

$a_1 - 1 + \frac{1}{n} < \frac{a_n}{n} < a_1 + 1 - \frac{1}{n}$

$a_1 - 1 < w < a_1 + 1 \Rightarrow na_1 - n < nw < na_1 + n \Rightarrow$

$a_m < ka_n + a_r + k =$

$\frac{m-r}{n} a_n + a_r + \frac{m-r}{n}$

$\frac{a_m}{m} < \frac{m-r}{m} \frac{a_n}{n} + \frac{a_r}{m} + \frac{m-r}{m} \frac{1}{n}$

$w <$ $\qquad$ $\frac{a_n}{n} + 0 + \frac{1}{n}$

$\left| w - \frac{a_m}{m} \right| = \left| \lim_{n \to \infty} \frac{a_{2^n m}}{2^n m} - \frac{a_m}{m} \right| = \left| \qquad - \frac{a_{2m}}{2m} + \frac{a_{2m}}{2m} - \frac{a_m}{m} \right|$

1.6.3

Friday, July 3, 2026 2:47 PM

$1 \dots N$

$\left(\frac{1}{d_1}+\dots+\frac{1}{d_n}\right)(N+1)-\left(\frac{a_1}{d_1}+\dots+\frac{a_n}{d_n}\right)+n \overset{N\to\infty}{\Longrightarrow} \frac{1}{d_1}+\dots+\frac{1}{d_n}\geq 1$

$\vee$

$\left\lceil\frac{N+1-a_1}{d_1}\right\rceil+\left\lceil\frac{N+1-a_2}{d_2}\right\rceil+\dots+\left\lceil\frac{N+1-a_n}{d_n}\right\rceil=N$

$\vee$

$\left(\frac{1}{d_1}+\dots+\frac{1}{d_n}\right)(N+1)-\left(\frac{a_1}{d_1}+\dots+\frac{a_n}{d_n}\right) \overset{N\to\infty}{\Longrightarrow} \frac{1}{d_1}+\dots+\frac{1}{d_n}\leq 1$

$a_1<a_1+d_1<\dots<\underline{a_1+kd_1}\leq N<\underline{N+1}\leq \underline{a_1+(k+1)d_1}$

$k<\frac{N+1-a_1}{d_1}$ $\qquad \frac{N+1-a_1}{d_1}\leq k+1$

$\frac{1}{d_1}+\dots+\frac{1}{d_n}=1 \Rightarrow d_2\cdots d_n+d_1d_3\cdots d_n+\dots+d_1\cdots d_{n-1}=d_1d_2\cdots d_n$

$\frac{d_2\cdots d_n}{d_1}+d_3\cdots d_n+\dots+d_2\cdots d_{n-1}=d_2\cdots d_n$

$\frac{d_n}{d_1}+\dots+\frac{d_n}{d_{n-1}}+1=d_n$

$\vee$

$\frac{d_n}{d_1}+n-1<d_n \Rightarrow d_n>\frac{n-1}{1-\frac{1}{d_1}}$

$d_i^2 \mid d_1\cdots d_n$

$f(z)=\frac{p'}{p}$ $\qquad Res(d_i)=1$

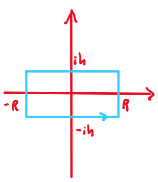
$p(x)=(x-d_1)\dots(x-d_n)=\sum_{k=0}^{n}\alpha_k x^k$

$0=p(d_n)=d_n^n-\alpha_{n-1}d_n^{n-1}+\dots$

$(-1)^n p'(0)=d_1\cdots d_{n-1}+\dots+d_2\cdots d_n$

$f(x)=\ln p,\quad f'=\frac{p'}{p},\quad f'(0)=-1$

$p(0)=p'(0)$



$2\pi i\sum Res(f,d_j)=\int\frac{p'}{p}dz$

$x+x^2+\dots=(x^{a_1}+x^{a_1+d_1}+\dots)+\dots+(x^{a_n}+x^{a_n+d_n}+\dots)$

$\frac{x}{1-x}=\sum_{i=1}^{n}\frac{x^{a_i}}{1-x^{d_i}}$

$x\prod_{i=1}^{n}(1-x^{d_i})=x^{a_1}(1-x)(1-x^{d_2})\dots(1-x^{d_n})+\dots$

$+x^{a_n}(1-x)(1-x^{d_1})\dots(1-x^{d_{n-1}})$

$e^{j\frac{2\pi}{d_n}k}$

$d_n \mid d_j$

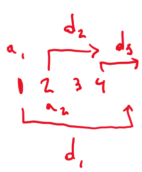
$e^{j\frac{2\pi}{d_n}d_j}=1 \Rightarrow$

$a_j+\sum_{j\neq i}d_i=a_j-d_j+\sum d_i$

$1+\sum d_i=a_j+1+\sum_{i\neq j}d_i$

$d_j=a_j$

$1+d_2+\dots+d_n$



$a_3\leq 4$

$a_4\leq$

$a_1=1$

$a_2=2$

$\vdots$

$a_{d-1}=d-1$

$a_d=d+1$

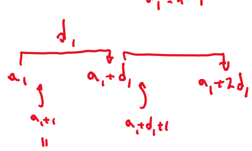
$N\equiv a_1 \pmod{d_1}$

$\vdots$

$N\equiv a_n \pmod{d_n}$

$d_1\leq n-1$

$d_1\leq n-1$



~~0 1 2 3 4 5 6 7 8 9 10 11~~

1.6.4

Sunday, July 5, 2026 3:36 PM

$$a_3 - a_2 = a_2 - a_1 \Rightarrow a_3 = 2a_2 - a_1$$

$$\frac{a_4}{a_3} = \frac{a_3}{a_2} \Rightarrow a_4 = \frac{a_3^2}{a_2} = \frac{(2a_2 - a_1)^2}{a_2} = 4a_2 - 4a_1 + \frac{a_1^2}{a_2}$$

$$a_5 - a_4 = a_4 - a_3 \Rightarrow a_5 = 2a_4 - a_3 = 8a_2 - 8a_1 + 2\frac{a_1^2}{a_2} - 2a_2 + a_1 = 6a_2 - 7a_1 + 2\frac{a_1^2}{a_2}$$

$$\frac{a_6}{a_5} = \frac{a_5}{a_4} \Rightarrow a_6 = \frac{a_5^2}{a_4} = \frac{(2a_4 - a_3)^2}{a_4} = 4a_4 - 4a_3 + \frac{a_3^2}{a_4} = 4\left(4a_2 - 4a_1 + \frac{a_1^2}{a_2} - 2a_2 + a_1\right) + a_2 = 9a_2 - 12a_1 + 4\frac{a_1^2}{a_2}$$

$$a_7 - a_6 = a_6 - a_5 \Rightarrow a_7 = 2a_6 - a_5 = 2\left(9a_2 - 12a_1 + 4\frac{a_1^2}{a_2}\right) - 6a_2 + 7a_1 - 2\frac{a_1^2}{a_2} = 12a_2 - 17a_1 + 6\frac{a_1^2}{a_2}$$

$$\frac{a_8}{a_7} = \frac{a_7}{a_6} \Rightarrow a_8 = \frac{a_7^2}{a_6} = \frac{(2a_6 - a_5)^2}{a_6} = 4a_6 - 4a_5 + \frac{a_5^2}{a_6} = 4\left(9a_2 - 12a_1 + 4\frac{a_1^2}{a_2} - 6a_2 + 7a_1 - 2\frac{a_1^2}{a_2}\right) + 4a_2 - 4a_1 + \frac{a_1^2}{a_2} = 16a_2 - 24a_1 + 9\frac{a_1^2}{a_2}$$

Guess: $a_{2k+1} = \left(2\sum_{i=1}^{k} i\right)a_2 - \left(2\sum_{i=1}^{k}(2i-1) - 1\right)a_1 + 2\sum_{i=1}^{k-1} i \, \frac{a_1^2}{a_2}$

$$a_{2k+2} = \left(2\sum_{i=1}^{k} i + k + 1\right)a_2 - \left(2\sum_{i=1}^{k}(2i-1) + 2k\right)a_1 + \left(2\sum_{i=1}^{k-1} i + k\right)\frac{a_1^2}{a_2}$$

$$a_{2k+3} = 2a_{2k+2} - a_{2k+1} = \left(2\sum_{i=1}^{k} i + 2(k+1)\right)a_2 + \left(-2\sum_{i=1}^{k}(2i-1) - 4k - 1\right)a_1 + \left(2\sum_{i=1}^{k-1} i + 2k\right)\frac{a_1^2}{a_2}$$

$$a_{2k+4} = \frac{a_{2k+3}^2}{a_{2k+2}} = \frac{(2a_{2k+2} - a_{2k+1})^2}{a_{2k+2}} = 4(a_{2k+2} - a_{2k+1}) + \frac{a_{2k+1}^2}{a_{2k+2}} = 4(a_{2k+2} - a_{2k+1}) + a_{2k} =$$

$$4\left((k+1)a_2 - (2k+1)a_1 + k\frac{a_1^2}{a_2}\right) + \left(2\sum_{i=1}^{k-1} i + k\right)a_2 - \left(2\sum_{i=1}^{k-1}(2i-1) + 2k - 2\right)a_1 + \left(2\sum_{i=1}^{k-2} i + k - 1\right)\frac{a_1^2}{a_2}$$

$$\left(2\sum_{i=1}^{k+1} i + k + 2\right)a_2 + \left(-2\sum_{i=1}^{k+1}(2i-1) + 1\right)a_1 + \left(2\sum_{i=1}^{k} i + k + 1\right)\frac{a_1^2}{a_2} \quad V$$

1.6.5

Thursday, July 2, 2026 6:33 PM

$$(a_m(j+k)+1) = (a_m(j)+1)^{2^k}$$

$$\Downarrow$$

$$a_n(n)+1 = (a_n(0)+1)^{2^n} = \left[\left(\frac{d}{2^n}+1\right)^{\frac{2^n}{d}}\right]^d$$

1.6.6

Sunday, July 5, 2026 4:25 PM

$$\tan(\varphi+\theta) = \frac{\overset{x}{\overset{\shortparallel}{\tan\varphi}} + \overset{y}{\overset{\shortparallel}{\tan\theta}}}{1-\tan\theta\tan\varphi}$$

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$$

$$\operatorname{arccot}\frac{1-xy}{x+y}$$

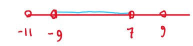
1.6.7

Sunday, July 5, 2026 6:03 PM

$|x+2| < 3^2$, $|x| < 3^2$ $\quad -9 < x < 7$

$$S_n = \sum_{k=1}^{3^{N+1}} \frac{x^{a_k}}{k^3} = \sum_{n=0}^{N} \sum_{3^n \le k < 3^{n+1}} \frac{x^{a_k}}{k^3} < \sum_{n=0}^{N} \sum_{\ell=0}^{n-1} \sum_{\substack{3^n \le k < 3^{n+1} \\ a_k = \ell}} \frac{x^{a_k}}{k^3} < \sum_{n=0}^{N} \frac{1}{3^{3n}} \sum_{\ell=0}^{n-1} \sum_{\substack{3^n \le k < 3^{n+1} \\ a_k = \ell}} x^{a_k} = \sum_{n=0}^{N} \frac{1}{3^{3n}} \sum_{\ell=0}^{n-1} \binom{n}{\ell} 2^{n-\ell} x^{\ell} = \sum_{n=0}^{N} \frac{1}{3^{3n}} \left[(x+2)^n - x^n \right] = \sum_{n=0}^{N} \frac{1}{3^{3n}} \left[(x+2)^n - x^n \right] = \sum_{n=0}^{N} \frac{(x+2)^n}{3^{3n}} - \sum_{n=0}^{N} \frac{x^n}{3^{3n}} =$$

$$> \sum_{n=0}^{N} \frac{1}{3^{3n+3}} \sum_{a_k=0}^{n-1} x^{a_k} = \sum_{n=0}^{N} \sum_{3^n \le k < 3^{n+1}} \frac{1}{3^{3n+3}} \sum_{\ell=0}^{n-1} \binom{n}{\ell} 2^{n-\ell} x^{\ell} = \sum_{n=0}^{N} \sum_{3^n \le k < 3^{n+1}} \frac{1}{3^{3n+3}} \left[(x+2)^n - x^n \right] = \sum_{n=0}^{N} \frac{2 \cdot 3^n}{3^{3n+3}} \left[(x+2)^n - x^n \right] = \frac{2}{27} \left(\sum_{n=0}^{N} \frac{(x+2)^n}{3^{2n}} - \sum_{n=0}^{N} \frac{x^n}{3^{2n}} \right) = \frac{2}{27} \left(\frac{1 - \left(\frac{x+2}{9}\right)^{n+1}}{1 - \frac{x+2}{9}} - \frac{1 - \left(\frac{x}{9}\right)^{n+1}}{1 - \frac{x}{9}} \right)$$



1.6.8

Sunday, July 5, 2026 6:30 PM

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \cos y_n & -\sin y_n \\ \sin y_n & \cos y_n \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \cos\left(\sum_{i=1}^{n-1} y_i\right) & -\sin\left(\sum_{i=1}^{n-1} y_i\right) \\ \sin\left(\sum_{i=1}^{n-1} y_i\right) & \cos\left(\sum_{i=1}^{n-1} y_i\right) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\sin\alpha - \sin\beta = \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha-\beta}{2} - \frac{\alpha+\beta}{2}\right) = \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} - \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} =$$

$$2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$A(\theta)A(\varphi) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{bmatrix} = A(\varphi+\theta)$$

$y_n \to y$

$$y_{n+1} - y_n = \left(\sin\sum_{i=1}^{n} y_i - \sin\sum_{i=1}^{n-1} y_i\right)x_1 + \left(\cos\sum_{i=1}^{n} y_i - \cos\sum_{i=1}^{n-1} y_i\right)y_1 = 2x_1\sin\frac{y_n}{2}\cos\left(\sum_{i=1}^{n-1} y_i + \frac{y_n}{2}\right) + 2y_1\sin\frac{y_n}{2}\sin\left(\sum_{i=1}^{n-1} y_i + \frac{y_n}{2}\right) = 2\sin\frac{y_n}{2}\cos\left(\sum_{i=1}^{n-1} y_i + \frac{y_n}{2} - \varphi\right)$$

$\cos\varphi = x_1$

$\sin\varphi = y_1$

$x_n = \cos\varphi_n, \; y_n = \sin\varphi_n$

$\cos\varphi_{n+1} = x_{n+1} = \cos(\varphi_n + y_n)$ $\quad -1 \le y_n \le 1 \Rightarrow -\frac{\pi}{2} \le \arcsin y_n = \varphi_n \le \frac{\pi}{2}$

$\sin\varphi_{n+1} = y_{n+1} = \sin(\varphi_n + y_n)$ $\quad -\frac{\pi}{2} - 1 \le \varphi_n + y_n \le \frac{\pi}{2} + 1 < \pi$

$\varphi_{n+1} = \varphi_n + \sin\varphi_n$

1.6.27

Tuesday, June 30, 2026 6:17 PM

$$\lim_{n\to\infty} \sum_{k=1}^{n^2} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \sum_{k=1}^{n^2} \frac{1}{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} = \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

$$\tan(\varphi+\theta) = \frac{\tan\varphi + \tan\theta}{1-\tan\theta\tan\varphi}$$

$$\frac{n}{n^2+k^2} \overset{\sim}{=} \arctan\frac{k+1}{n} - \arctan\frac{k}{n} = \arctan\frac{\frac{k+1}{n}-\frac{k}{n}}{1+\frac{k}{n}\frac{k+1}{n}} = \arctan\frac{n}{n^2+k(k+1)}$$

$\tan\theta > \theta$, $0<\theta<\frac{\pi}{2}$

- $\frac{n}{n^2+k^2} \geq \arctan\frac{n}{n^2+k^2} > \arctan\frac{n}{n^2+k(k+1)}$
- $\arctan\frac{n}{n^2+k(k+1)} > \frac{n(n^2+k(k+1))}{n^2+(n^2+k(k+1))^2} \overset{?}{>} \frac{n}{n^2+k^2} - O(n^2)$

$\frac{x}{x^2+1} < \arctan x$ $\quad x = \tan\theta$

$\frac{\tan\theta}{\tan^2\theta+1} < \theta \iff \sin\theta\cos\theta < \theta \iff \sin 2\theta < 2\theta$ ✓

NA 1-5

Monday, July 6, 2026 11:18 AM

$16 \to 2003\cdot 3+1 = 6010$

$1, 4, \ldots : 2004-5=1999$

$9, 16, \ldots : \left\lceil \frac{6010+1-16}{7} \right\rceil = \left\lceil \frac{5995}{7} \right\rceil = 857 \quad - \left\lceil \frac{6010+1-16}{21} \right\rceil =$

$n_1+n_2+n_3+n_4+n_5+n_6 = 79 \qquad n_1 = 1$

$n_2+n_3+n_4+n_5+n_6 = 78 \qquad n_2 \mid 78 = 2\cdot 39$

$\geq 15 n_2 \qquad n_2 = 2$

$n_3+n_4+n_5+n_6 = 76 \qquad n_3 \mid 76 = 2^2\cdot 19$

$\geq 10 n_3 \qquad n_3 = 4$

$n_4+n_5+n_6 = 72 \qquad n_4 \mid 72 = 2^3\cdot 3^2$

$\geq 6 n_4 \qquad n_4 \leq 12 \qquad n_4 = 8, 12$

- $n_4 = 8$

$n_5+n_6 = 64 \qquad n_5 \mid 64 = 2^6$

$\geq 3 n_5 \qquad n_5 \leq \frac{64}{3} \Rightarrow n_5 \leq 21$

$n_5 = 16, \quad n_6 = 48$

- $n_4 = 12$

$n_5+n_6 = 60 \qquad n_5 \mid 60 = 2^2\cdot 3\cdot 5$

$n_5 \leq 20$

$n \equiv -10 \pmod{n+10} \Rightarrow$

$n^3 \equiv -1000 \pmod{n+10} \Rightarrow$

$900 \equiv 0 \pmod{n+10} \Rightarrow n+10 \mid 900$

$\mathbb{N}^* \ni \frac{n^3+100}{n+10} = \frac{n^3+10n^2-10n^2-100n+100n-1000+900}{n+10} =$

$n^2-10n+100+\frac{900}{n+10}$

$\gcd(30n+2, 12n+1) = \gcd(30n+2-2\cdot(12n+1), 12n+1) =$

$\gcd(6n, 12n+1) = \gcd(6n, 12n+1-2\cdot 6n) = \gcd(6n, 1) = 1$

$\gcd(a, b) \overset{?}{=} \gcd(a-b, b)$

$d \mid a, b \Rightarrow d \mid a-b, b \Rightarrow d \leq \gcd(a-b, b)$

$d \mid a-b, b \Rightarrow d \mid a, b \Rightarrow d \leq \gcd(a, b)$

$a_1(a_2a_3+a_3a_5+a_4a_5+a_2a_4) + a_6(a_2a_3+a_3a_5+a_4a_5+a_2a_4) =$

$(a_1+a_6)(a_2+a_5)(a_3+a_4)$

NA 6-10

Monday, July 6, 2026 12:31 PM

$999-\left(\left\lfloor\frac{999}{2}\right\rfloor+\left\lfloor\frac{999}{3}\right\rfloor+\left\lfloor\frac{999}{5}\right\rfloor-\left\lfloor\frac{999}{6}\right\rfloor-\left\lfloor\frac{999}{10}\right\rfloor-\left\lfloor\frac{999}{15}\right\rfloor+\left\lfloor\frac{999}{30}\right\rfloor\right)-168+3-1$

$S=p(1)+..+p(9)=1+2+..+9=45$

$p(10)+..+p(99)=1(1+2+..+9)+2(1+..+9)+..+9(1+..+9)=s^2$

$p(100)+..+p(999)=1(p(0)+p(1)+..+p(99))+..+9(p(9)+..+p(99))=s(s^2+s)$

$p(1)+..+p(999)=s(s+1)^2$

$lcm(m,n)+gcd(m,n)=m+n \Rightarrow mn+gcd^2(m,n)=m+n \Rightarrow (m-gcd(m,n))(n-gcd(m,n))=0$

$n=x\,gcd(m,n),\ m=y\,gcd(m,n) \quad (x,y)=1$

$a=p_1^{\alpha_1}..p_k^{\alpha_k},\ b=p_1^{b_1}..p_k^{b_k}$

$d(n^2)=(2\cdot31+1)(2\cdot19+1)$

$(\#\text{ divisors} > n)=\frac{d(n^2)-1}{2}$

$1,..,4012$

$2^{11}:\left\lfloor\frac{4012}{2^{11}}\right\rfloor=1$

$2^{10}:\left\lfloor\frac{4012}{2^{10}}\right\rfloor=3 \quad 1+3$

$2^{9}:\left\lfloor\frac{4012}{2^{9}}\right\rfloor=7 \quad 1+3+5+7$

$2^{8}:\left\lfloor\frac{4012}{2^{8}}\right\rfloor=15 \quad 1+3+5+..+15$

$\sum_{cyc}(1+2+2^2+2^3)(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3})(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3})$

$+\sum$

$\sum\frac{1}{2^i3^j5^k}+\sum 2^i3^j5^k$

$\parallel$

$(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3})(1+..+\frac{1}{3^3})(1+..+\frac{1}{5^3})+(1+..+2^3)(1+..+3^3)(1+..+5^3)$

$=(\frac{1}{2^3}+..+2^3)(\frac{1}{3^3}+..+3^3)(\frac{1}{5^3}+..+5^3)$

- $a=2^3\cdot5^3,\ b\to16$ cases

NA 11-15

Monday, July 6, 2026 2:40 PM

$p^{\alpha} | a,\ p^{\beta} | b,\ p^{\gamma} | c,\ \alpha \leq \beta \leq \gamma$

$$\frac{p^{2\gamma}}{p^{\beta} p^{\gamma} p^{\gamma}} = \frac{p^{2\alpha}}{p^{\alpha} p^{\alpha} p^{\beta}}$$

$$\operatorname{lcm}(\alpha, \beta, \gamma) = \frac{\operatorname{lcm}(\alpha, \beta) \cdot \gamma}{\gcd(\operatorname{lcm}(\alpha, \beta), \gamma)}$$

$\frac{1}{x} - \frac{1}{y} = \frac{1}{z} \Leftrightarrow z(y-x) = xy \Rightarrow z'(y - d_1 x') = x' y \Rightarrow \quad d_2 y' - d_1 x' = x' y'$

$y' | x'$

$x = d x'$

$y = d y' \qquad \gcd(x, y, z) = 1 \qquad x = d_1 x' \quad \gcd(x', z') = 1 \quad y = d_2 y'$

$z = d z' \qquad\qquad\qquad\qquad z = d_1 z' \qquad\qquad z' = d_2 z''$

$a^{3k+1} \equiv b^{3k+1} \equiv 1 \pmod p$

$hxyz = h^4 d_1^2 d_2^2 (x')^2$

$p | a^2 + ab + b^2 \Rightarrow a^3 \equiv b^3 \pmod p \Rightarrow a^{3k} \equiv b^{3k} \pmod p \Rightarrow a^2 \equiv b^2 \pmod p \Rightarrow p | (a-b)(a+b) \Rightarrow \begin{cases} a \equiv b \pmod p \Rightarrow (a-b)^2 \equiv 0 \pmod p \Rightarrow p | 3ab \\ (a+b)^2 \equiv 0 \pmod p \Rightarrow ab \equiv 0 \pmod p \end{cases}$

$a^{-1} \equiv b^{-1} \pmod p \Rightarrow a \equiv b \pmod p$

NA 16-20

Tuesday, July 7, 2026 4:29 PM

$$27000001 = 300^3+1 = 301(300^2-300+1) = 301(300^2+2\cdot 300+1-900) = 301(301^2-30^2) = 301\cdot 331\cdot 271 = 7\cdot 43\cdot 331\cdot 271$$

$$\binom{p-1}{k+1} = \frac{(p-1)!}{(p-k-2)!(k+1)!} = \frac{(p-1)!}{(p-1-k)!\,k!}\quad \frac{p-k-1}{k+1} = \binom{p-1}{k}(-k-1)\alpha \equiv -\binom{p-1}{k}$$

$$\frac{A}{B} \in \mathbb{N}^*$$

$$\alpha(k+1) \equiv 1 \pmod p$$

$$\frac{A}{B} \equiv A\cdot B^{-1} \pmod p \Leftrightarrow \left(\frac{A}{B}\right)B \equiv (A\cdot B^{-1})B \pmod p \Rightarrow \frac{AB}{B} \equiv A \pmod p$$

$$2^n - n = 2^{(p-2)p^n+1} - [(p-2)p^n+1] \equiv 2^{-1+1} - 1 = 2^0 - 1 = 0$$

$$n = k(p-1) \qquad k = lp-1$$

$p<n$

$$1+2!+\dots+n! \equiv 1+\dots+(p-1)! \equiv 1+2!+\dots+\left(\frac{p-1}{2}\right)! + \left(\frac{p+1}{2}\right)! + \dots + \underbrace{(p-3)!}_{-2^{-1}} + \underbrace{(p-2)!}_{1} + \underbrace{(p-1)!}_{-1}$$

$(3!)^{-1}$

$$m^k = m^{k'p} = (m^{k'})^p \qquad 1+2!+\dots+n! \le 1+n+n^2+\dots+n^n < \frac{n^{n+1}}{n-1}$$

NA 26-30

Monday, July 6, 2026 2:36 PM

$$\sigma(1)+\sigma(2)+\dots+\sigma(n)=\sum_{i=1}^{n} i\,\left|\{1\le k\le n : i|k\}\right| = \sum_{i=1}^{n} i\left\lfloor \frac{n}{i}\right\rfloor \le \sum_{i=1}^{n} i\,\frac{n}{i} = \sum_{i=1}^{n} n = n^2$$

NA 31-35

Wednesday, July 8, 2026 5:56 PM

$p_1 \cdots p_n = 10(p_1 + \dots + p_n) \Rightarrow p_3 \cdots p_n = 7 + p_3 + \dots + p_n$

• $n \geq 5$ $\quad p_3 \leq \dots \leq p_n \quad 2^{n-3} p_n \leq p_3 \cdots p_n = 7 + p_3 + \dots + p_n \leq 7 + (n-2)p_n \Rightarrow$

$(2^{n-3} - n + 2)p_n \leq 7$ thus $n < 6$

$n=3 \quad p_3 = 7 + p_3$

$n=4 \quad p_3 p_4 = 7 + p_3 + p_4 \Rightarrow (p_3 - 1)(p_4 - 1) = 8 \Rightarrow$ $\quad p_3 - 1 = 1,\ p_4 - 1 = 8$ $\quad p_3 - 1 = 2,\ p_4 - 1 = 4$

$n=5 \quad p_3 p_4 p_5 = 7 + p_3 + p_4 + p_5$

• $p_5 = 2$

$p_5 = 3 \quad 3p_3 p_4 = 10 + p_3 + p_4$

$p_5 = 5 \quad 5p_3 p_4 = 12 + p_3 + p_4 \quad (p_3 - 1)(p_4 - 1) \geq 1 \Rightarrow p_3 p_4 \geq p_3 + p_4$

$4p_3 p_4 \leq 12 \Rightarrow p_3 p_4 \leq 3$

$p_5 = 7 \quad 7p_3 p_4 = 14 + p_3 + p_4$

interesting: $\alpha_4 \alpha_3 \alpha_2 \alpha_1 \alpha_0 \cdot 99999 =$

$$\begin{array}{r} \alpha_4 \alpha_3 \alpha_2 \alpha_1 \alpha_0 00000 \\ - \qquad \alpha_4 \alpha_3 \alpha_2 \alpha_1 \alpha_0 \\ \hline \alpha_4 \alpha_3 \alpha_2 (\alpha_1 - 1)(9)(9-\alpha_4) \dots (10 - \alpha_2)(0) \end{array}$$

$\dfrac{10!}{5!} - \dfrac{9!}{5!}$

$k+1 = 2m, \quad 2k$

$\gcd(2m, 2k) = 2\gcd(m, k) = 2$

$\dfrac{2m + 2k}{2} = m + k$

$m+1$

$\boxed{9\overset{2}{8}7 \cdot 6}$

$0 \leftrightarrow 9$

$0 \leftrightarrow 0$

$x \cdot 11^n \mid x_n \quad \ell_n = 2 \cdot 11^n$

$x_{n+1} = \overbrace{\overline{x_n \cdots x_n}}^{11} = x_n \cdot 1\underbrace{0..0}_{\ell_n}10..01 \quad \ell_{n+1} = 2 \cdot 11^{n+1}$

NA 36-40

Thursday, July 9, 2026 3:59 PM

$a_1 = a + ab = a(b+1)$

$a_2 = a + (a^2 + ca)b = a(1 + ab + cb)$

$1 + ab + cb = (b+1)^2 \Leftrightarrow a + c = b + 2 \Leftrightarrow c = b + 2 - a$

$1 + ab + cb = (b+1)^3 \Leftrightarrow a + c = b^2 + 3b + 3 \Leftrightarrow c = b^2 + 3b + 3 - a$

$\gcd(a + c^n b, a + c^m b) = \gcd(a + c^n b, bc^n(c^{m-n} - 1))$

$\gcd(n^a - 1, n^b - 1) = \gcd[(n^d)^y - 1, (n^d)^x - 1] =$

$(n^d - 1)\gcd(1 + n^d + \dots + n^{d(y-1)}, 1 + \dots + n^{d(x-1)})$

$a_1 = a + b \quad (a, a_1) = 1$

$a_2 = a + a_1 b$

$\gcd(a_k, a_n) = \gcd(a_k, a + a_k(\dots))$

$\gcd(n^a - 1, n^b - 1) = \gcd(n^a - 1, n^b - n^a) = \gcd(n^a - 1, n^a(n^{b-a} - 1)) =$

$\gcd(n^a - 1, n^{b-a} - 1)$

$a_{n+1} = a + a_1 \dots a_n b$

$a_k = a + a_1 \dots a_{k-1} b$

$\gcd(n^a + 1, n^b + 1) = \gcd(n^a + 1, n^a(n^{b-a} - 1)) = \gcd(n^a + 1, n^{b-a} - 1) =$

- $b - a \geq a$ $\quad \gcd(n^a + 1, n^{b-a} - 1 - n^{b-a} - n^{b-2a}) = \gcd(n^a + 1, n^{b-2a} + 1)$
- $b - a < a$ $\quad \gcd(n^a + 1 - n^a + n^{2a-b}, n^{b-a} - 1) = \gcd(n^{2a-b} + 1, n^{b-a} - 1)$

$\left.\begin{matrix} k = n + m \\ l = m - n \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} m = \frac{k+l}{2} \\ n = \frac{k-l}{2} \end{matrix}\right\}$

$\gcd(n^d + 1, 2), \quad \gcd(n^d - 1, 2) = \gcd(n^d + 1, 2)$

$(5^n + 7^n)(5^m + 7^m) = 5^{n+m} + 7^{n+m} + 5^n 7^n (5^{m-n} + 7^{m-n})$

$\gcd(5^m + 7^m, 5^n + 7^n) = \gcd(5^m - 5^n 7^{m-n}, 5^n + 7^n) = \gcd(5^n + 7^n, 7^{m-n} - 5^{m-n})$

- $m - n \geq n$ $\quad \gcd(5^n + 7^n, 7^{m-n} - 5^{m-n} - 7^{m-2n}(5^n + 7^n)) = \gcd(5^n + 7^n, 5^n(+5^{m-2n} + 7^{m-2n}))$
- $m - n < n$ $\quad \gcd(5^n + 7^n - 7^{2n-m}(7^{m-n} - 5^{m-n}), 7^{m-n} - 5^{m-n}) = \gcd(5^{m-n}(5^{2n-m} + 7^{2n-m}), 7^{m-n} - 5^{m-n})$

$\underbrace{\gcd(5^d + 7^d, 2)}_{=2}$ or $\gcd(2, 7^d - 5^d) = 2$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n}{k} - \binom{n}{k-1} \right]^2 \overset{?}{=} \frac{1}{n+1}\binom{2n}{n}$$

$$\binom{n-1}{k-1}^2$$

$$\frac{1}{n+1}\binom{2n}{n} = \left(\int_0^x (1+t)^{2n}\,dt\right)[x^{n+1}] = \left(\frac{1}{2n+1}(1+t)^{2n+1}\right)[x^{n+1}] = \left(\frac{1}{2n+1}(1+t)^{3}\underbrace{(1+t)^{n-1}}_{\sum_{k=0}^{n-1}\binom{n-1}{k}t^k}\underbrace{(1+t)^{n-1}}_{\sum_{k=0}^{n-1}\binom{n-1}{n-1-k}t^{n-1-k}}\right)[x^{n+1}] =$$

$$x^{n-1} \sum_{k=0}^{n-1}\binom{n-1}{k}^2$$

$$\left(\frac{1}{2n+1}(1+t)^3\right.$$

$x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{2n}$

$S = \{(x, y_1, \dots, y_{n-1}) : x \in \{x_1, \dots, x_n\},\ y_i \in \{x_1, \dots, x_{2n}\} \setminus \{x\},\ y_i \neq y_j\ \forall i \neq j\}$

$S_k = \{(x, u_1, \dots, u_k, v_1, \dots, v_{n-k}) : u_i \in \{x_1, \dots, x_n\},\ u_i \neq u_j,\ v_i \in \{x_{n+1}, \dots, x_{2n}\},\ v_i \neq v_j,\ x \in \{u_1, \dots, u_k\}\}$

$|S_k| = \binom{n}{k}\binom{n}{n-k} k = k\binom{n}{k}^2$

$|S| = n\binom{2n-1}{n-1}$

$f : \bigcup_{k=0}^{n} S_k \to S$ $\quad$ $S_i \cap S_j \neq \emptyset$ ($i<j$) $\quad$ $(x, u_1, \dots, u_i, v_1, \dots, v_{n-i}) = (x', u_1', \dots, u_j', v_1', \dots, v_j') \Rightarrow v_1' = u_{i+1}$

$f((x, (u_1, \dots, u_k), (v_1, \dots, v_{n-k}))) = (x, (u_1, \dots, u_k, v_1, \dots, v_{n-k}))$

$(x, (y_1, \dots, y_{n-1})) = f((x, (\underline{y_1, \dots, y_j}), (y_{j+1}, \dots, y_{n-1}))$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n}{k} - \binom{n}{k-1} \right]^2 = \underbrace{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}^2}_{0 \le k \le \lfloor \frac{n}{2} \rfloor} - 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}\binom{n}{k-1} + \underbrace{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k-1}^2}_{\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{n-k+1}^2}$$

$\cdot\ n = 2m$

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

$$2\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{k}\binom{n}{k-1} = 2\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{k}\binom{n}{n-k+1} = \sum_{k=1}^{n}\binom{n}{k}\binom{n}{n-k+1} = \binom{2n}{n+1}$$

$1 \le k \le \frac{n}{2}$ $\quad$ $n-k+1 \le n$, $\frac{n}{2}+1$

$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n}$$

$\frac{n-1}{2}$

$n - \lfloor \frac{n}{2} \rfloor + 1 \le n-k+1 \le n$

$n - \frac{n-1}{2} + 1 = \frac{n+3}{2}$

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n-1}{k-1}^2 = \frac{1}{2}\sum_{k=1}^{n}\binom{n-1}{k-1}^2 = \frac{1}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 = \frac{1}{2}\binom{2n-2}{n-1} = \frac{1}{2}\,\frac{n^2}{2n(2n-1)}\binom{2n}{n}$$

$\binom{n-1}{k-1} = \binom{n-1}{n-k}$ $\quad$ $n - \lfloor \frac{n}{2} \rfloor \le n-k \le n-1$, $n - \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$