

Lemma 1.3

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Denote the eigenvalues of A by $\lambda_i, i=1,2,\dots,n$. Assume $\exists i_0 \in \{1,\dots,n\} : \lambda_{i_0} \neq 0$

Without loss of generality assume $\lambda_1, \lambda_2, \dots, \lambda_m$ are the only non-zero eigenvalues.

Then $\lambda_1, \lambda_2, \dots, \lambda_m \neq 0, \lambda_{m+1} = \dots = \lambda_n = 0$ and $\text{tr}(A^k) = 0 \Rightarrow \sum_{i=1}^n \lambda_i^k = 0 \Rightarrow \sum_{i=1}^m \lambda_i^k = 0 \quad \forall k \geq k_0$

Suppose $k_0 \geq 2$ and consider $p(\lambda) = \lambda^{k_0-1}(\lambda - \lambda_1) \dots (\lambda - \lambda_m) = \sum_{j=0}^m \alpha_j \lambda^{j+k_0-1}$

then $p(\lambda_i) = 0 \quad \forall i \in \{1, \dots, m\} \Rightarrow \sum_{i=1}^m p(\lambda_i) = 0 \Rightarrow \sum_{i=1}^m \sum_{j=0}^m \alpha_j \lambda_i^{j+k_0-1} = 0 \Rightarrow \sum_{j=0}^m \sum_{i=1}^m \alpha_j \lambda_i^{j+k_0-1} = 0 \Rightarrow$

$$\sum_{j=0}^m \alpha_j \sum_{i=1}^m \lambda_i^{j+k_0-1} = 0 \Rightarrow \alpha_0 \sum_{i=1}^m \lambda_i^{k_0-1} + \sum_{j=1}^m \alpha_j \sum_{i=1}^m \lambda_i^{j+k_0-1} = 0 \xRightarrow{j+k_0-1 \geq k_0} \alpha_0 \sum_{i=1}^m \lambda_i^{k_0-1} + \sum_{j=1}^m \alpha_j \cdot 0 = 0 \Rightarrow$$

$$\alpha_0 \sum_{i=1}^m \lambda_i^{k_0-1} = 0,$$

but α_0 is the constant term of the polynomial $(\lambda - \lambda_1) \dots (\lambda - \lambda_m)$,

hence $\alpha_0 = (-1)^m \lambda_1 \dots \lambda_m \neq 0$ thus $\sum_{i=1}^m \lambda_i^{k_0-1} = 0$

We conclude by induction that $\sum_{i=1}^m \lambda_i^k = 0 \quad \forall k \geq 1 (=k_0-1 \text{ for } k_0=2)$

Repeating the argument one last time $\alpha_0 \sum_{i=1}^m \lambda_i^{1-1} = 0 \Rightarrow \alpha_0 \cdot m = 0 \Rightarrow \alpha_0 = 0$ contradiction

Therefore all eigenvalues of A are 0.

Lemma 1.4

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Take $k \in \mathbb{N}^*$, then $Q(A) = P^{k-1}(A)$ is a polynomial in A and thus $A, Q(A)$ commute.

$$\text{tr}(P^k(A)) = \text{tr}(P(A)Q(A)) = \text{tr}[(AB-BA)Q(A)] = \text{tr}(ABQ(A)) - \text{tr}(\underline{BAQ(A)}) =$$

$$\text{tr}(ABQ(A)) - \text{tr}(\underline{BQ(A)A}) = \text{tr}(ABQ(A)) - \text{tr}(ABQ(A)) = 0$$

Now using Lemma 1.3 $\text{tr}(P^k(A)) = 0 \ \forall k \geq 1 \Rightarrow 0$ is the only eigenvalue of $P(A) = AB - BA$

Problem 1

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Using Lemma 1.5 $A_0 = cI_2 \quad \forall c \in \mathbb{R} \implies A_k = a_k A_0 + b_k I_2, \quad a_k, b_k \in \mathbb{R}, \quad k \in \{1, \dots, n\}$

• If $a_k = 0 \quad \forall k \in \{1, \dots, n\}$, $\det\left(\sum_{k=1}^n A_k^2\right) = \det\left(\sum_{k=1}^n (b_k I_2)^2\right) = \det\left(\sum_{k=1}^n b_k^2 I_2\right) = \left(\sum_{k=1}^n b_k^2\right)^2 \det I_2 = \left(\sum_{k=1}^n b_k^2\right)^2 \geq 0$

• If $a_k \neq 0$ for some k , $\det\left(\sum_{k=1}^n A_k^2\right) = \det\left(\sum_{k=1}^n (a_k A_0 + b_k I_2)^2\right) = \det\left(\left(\sum_{k=1}^n a_k^2\right) A_0^2 + 2\left(\sum_{k=1}^n a_k b_k\right) A_0 + \left(\sum_{k=1}^n b_k^2\right) I_2\right)$

Let λ_1, λ_2 be the eigenvalues of A_0 , then since $A_0 \in M_2(\mathbb{R})$, $\chi_{A_0} \in \mathbb{R}[x]$, hence $\lambda_1, \lambda_2 \in \mathbb{R}$ or $\bar{\lambda}_1 = \lambda_2$.

Define $p(x) = \left(\sum_{k=1}^n a_k^2\right) x^2 + 2\left(\sum_{k=1}^n a_k b_k\right) x + \sum_{k=1}^n b_k^2$

Then $\det\left(\sum_{k=1}^n A_k^2\right) = \det(p(A_0)) = p(\lambda_1)p(\lambda_2)$

but $\sum_{k=1}^n a_k^2 > 0$, $\Delta = 4\left(\sum_{k=1}^n a_k b_k\right)^2 - 4\sum_{k=1}^n a_k^2 \cdot \sum_{k=1}^n b_k^2 \leq 0$ by the Cauchy-Schwarz inequality

$\exists r \in \mathbb{R} \quad p(x) = \alpha(x-r)(x-\bar{r})$, where $\alpha = \sum_{k=1}^n a_k^2 \geq 0$

So far $\det\left(\sum_{k=1}^n A_k^2\right) = p(\lambda_1)p(\lambda_2) = \alpha^2(\lambda_1-r)(\lambda_1-\bar{r})(\lambda_2-r)(\lambda_2-\bar{r})$

• If $\lambda_1, \lambda_2 \in \mathbb{R}$ $\det\left(\sum_{k=1}^n A_k^2\right) = \alpha^2(\lambda_1-r)(\lambda_1-\bar{r})(\lambda_2-r)(\lambda_2-\bar{r}) = \alpha^2(\lambda_1-r)(\lambda_1-r)(\lambda_2-r)(\lambda_2-r) = \alpha^2|\lambda_1-r|^2|\lambda_2-r|^2 \geq 0$

• If $\bar{\lambda}_1 = \lambda_2$ $\det\left(\sum_{k=1}^n A_k^2\right) = \alpha^2(\lambda_1-r)(\lambda_1-\bar{r})(\lambda_2-r)(\lambda_2-\bar{r}) = \alpha^2(\lambda_1-r)(\lambda_1-\bar{r})(\bar{\lambda}_1-r)(\bar{\lambda}_1-\bar{r}) = \alpha^2(\lambda_1-r)(\lambda_1-\bar{r})(\overline{\lambda_1-r})(\overline{\lambda_1-\bar{r}}) = \alpha^2|\lambda_1-r|^2|\lambda_1-\bar{r}|^2 \geq 0$

Problem 2

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$$\bullet \operatorname{tr}((X-Y)(X-Y+I_n)) = \operatorname{tr}(\underbrace{X^2 - YX - XY + Y^2}_{\text{trace is cyclic}} + X - Y) = \operatorname{tr}(2(XY + I_n) - YX - XY) = \operatorname{tr}(XY - YX + 2I_n) = \operatorname{tr}(XY) - \operatorname{tr}(YX) + 2\operatorname{tr}I_n = 2n$$

$$\bullet \text{ Let } Z = X - Y \Leftrightarrow X = Y + Z,$$

$$X^2 + Y^2 + X - Y = 2(XY + I_n) \Rightarrow (Y + Z)^2 + Y^2 + Z = 2(Y + Z)Y + 2I_n \Rightarrow Y^2 + Z^2 + YZ + ZY + Z = 2Y^2 + 2ZY + 2I_n \Rightarrow ZY - YZ = Z^2 + Z - 2I_n$$

Using Lemma 1.4, the eigenvalues of $Z^2 + Z - 2I_n$ are all 0, hence $\chi_{Z^2 + Z - 2I_n}(x) = x^n$

$$\det((X-Y)(X-Y+I_n)) = \det(Z(Z+I_n)) = \det(Z^2 + Z) = \det(2I_n + Z^2 + Z - 2I_n) = (-1)^n \det(-2I_n - (Z^2 + Z - 2I_n)) =$$

$$(-1)^n \chi_{Z^2 + Z - 2I_n}(-2) = (-1)^n (-2)^n = 2^n$$

Problem 4

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$$\begin{aligned}\text{Frobenius : } \text{rank}(A(A-B)) + \text{rank}((A-B)B) &\leq \text{rank}(A(A-B)B) + \text{rank}(A-B) \Rightarrow \\ \text{rank}(A^2 - AB) + \text{rank}(AB - B^2) &\leq \text{rank}(A^2B - AB^2) + \text{rank}(A-B) \Rightarrow \\ \text{rank}(A-AB) + \text{rank}(AB-B) &\leq \text{rank}(AB-AB) + \text{rank}(A-B) \Rightarrow \\ \text{rank}(A-AB) + \text{rank}(AB-B) &\leq \text{rank}(0) + \text{rank}(A-B) \Rightarrow \\ \text{rank}(A-AB) + \text{rank}(AB-B) &\leq \text{rank}(A-B) \Rightarrow\end{aligned}$$

$$\text{Also } \text{rank}(A-AB) + \text{rank}(AB-B) \geq \text{rank}(A-AB+AB-B) = \text{rank}(A-B)$$

$$\text{Therefore } \text{rank}(A-AB) + \text{rank}(AB-B) = \text{rank}(A-B)$$

Problem 8

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$$A^2 = \begin{pmatrix} O_n & B \\ B^T & O_m \end{pmatrix} \begin{pmatrix} O_n & B \\ B^T & O_m \end{pmatrix} = \begin{pmatrix} BB^T & O_{n \times m} \\ O_{m \times n} & B^T B \end{pmatrix}$$

Suppose $n \geq m$

$$\chi_{A^2}(\lambda) = \det(\lambda I_{n+m} - A^2) = \det \begin{pmatrix} \lambda I_n - BB^T & O_{n \times m} \\ O_{m \times n} & \lambda I_m - B^T B \end{pmatrix} = \det(\lambda I_n - BB^T) \det(\lambda I_m - B^T B) =$$

$$\chi_{BB^T}(\lambda) \chi_{B^T B}(\lambda) = \lambda^{n-m} \chi_{B^T B}^2(\lambda) \quad (*)$$

$B^T B$ symmetric and in fact positive semidefinite, hence its eigenvalues are positive or 0. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A . Then $(*)$ implies

$\lambda_i^4 = 0 \Leftrightarrow \lambda_i = 0$ (because of the term λ^{n-m}) or

$\lambda_i^2 = \mu_i \Leftrightarrow \lambda_i = \pm \sqrt{\mu_i}$ where $\mu_i > 0$ is an eigenvalue of $B^T B$.

Similarly for $n < m$.