

Math Competition Preparation Seminar

Linear Algebra

April 24, 2026

1 Theory/Background

1.1 Theorem

Let $n, m \in \mathbb{N}^*$: $n \geq m$ and $A \in \mathcal{M}_{n \times m}(\mathbb{C}), B \in \mathcal{M}_{m \times n}(\mathbb{C})$. Then $\chi_{AB}(x) = x^{n-m}\chi_{BA}(x)$, where χ_C denotes the characteristic polynomial of the (square) matrix C .

1.2 Frobenius Inequality

Let $n, m, k, l \in \mathbb{N}^*$ and $A \in \mathcal{M}_{n \times m}(\mathbb{C}), B \in \mathcal{M}_{m \times k}(\mathbb{C}), C \in \mathcal{M}_{k \times l}(\mathbb{C})$. Then

$$\text{rank}(AB) + \text{rank}(BC) \leq \text{rank}(ABC) + \text{rank}(B)$$

1.3 Lemma

Let $n \in \mathbb{N}$ and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\text{tr}(A^k) = 0, \forall k \geq k_0$ for some $k_0 \in \mathbb{N}^*$ then all eigenvalues of A are 0.

1.4 Lemma

Let $n \in \mathbb{N}^*$ and $A, B \in \mathcal{M}_n(\mathbb{C})$ satisfying $AB - BA = P(A)$, where P is a polynomial. Then the eigenvalues of $AB - BA$ are 0.

1.5 Lemma

Let $A, B \in \mathcal{M}_2(\mathbb{R})$: $AB = BA, A \neq cI_2, \forall c \in \mathbb{R}$. Then $\exists a, b \in \mathbb{R} : B = aA + bI_2$

2 Problems

Problem 1 Let $n \in \mathbb{N}^*$ and $A_0, A_1, \dots, A_n \in \mathcal{M}_2(\mathbb{R})$: $A_0 A_k = A_k A_0, \forall k \in \{1, 2, \dots, n\}$, $A_0 \neq cI_2, \forall c \in \mathbb{R}$. Show that $\det(\sum_{k=1}^n A_k^2) \geq 0$.

Problem 2 Let $n \in \mathbb{N}^*$ and $X, Y \in \mathcal{M}_n(\mathbb{C})$ such that $X^2 + Y^2 + X - Y = 2(XY + I_n)$. Calculate the following quantities:

- $\text{tr}((X - Y)(X - Y + I_n))$
- $\det((X - Y)(X - Y + I_n))$

Problem 3 Let $A, B \in \mathcal{M}_2(\mathbb{R})$, such that B is invertible and satisfying:

i) $A^2 + B^2 = 2AB$

ii) $\det(A^d - I_2) = \det(A^d + I_2)$, $d \in \{2020, 2021\}$

Prove that $\det(A^{2022} - I_2) = \det(A^{2022} + I_2)$.

Problem 4 Let $n \in \mathbb{N}^*$ and $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $A^2 = A, B^2 = B$. Prove that

$$\text{rank}(A - AB) + \text{rank}(AB - B) = \text{rank}(A - B)$$

Problem 5 Let $n \in \mathbb{N}^*$ and $A, B \in \mathcal{M}_n(\mathbb{C})$ satisfying

$$A = AB - BA + A^2B - 2ABA + BA^2 + A^2BA - ABA^2$$

Prove that $\det(A) = 0$.

Problem 6 Let $n \in \mathbb{N}^*$ and $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $A + B$ is an invertible matrix that commutes with A or B and $\text{rank}A + \text{rank}B = n$.

Prove that $A^k + B^k$ is invertible and $\text{tr}(A + B)^k = \text{tr}A^k + \text{tr}B^k$ for all $k \in \mathbb{N}^*$.

Problem 7 Let $n \in \mathbb{N}^*$ and $A, B \in \mathcal{M}_n(\mathbb{R})$ be symmetric matrices such that

$$(AB + BA - A^2 - I_n)^2 = AB^2 - B^2A$$

Find $\text{rank}(A^2 + B^2)$.

Problem 8 Let $n, m \in \mathbb{N}^*$ and $B \in \mathcal{M}_{n \times m}(\mathbb{C})$. Define

$$A = \begin{pmatrix} \mathcal{O}_n & B \\ B^T & \mathcal{O}_m \end{pmatrix}$$

Prove that the eigenvalues of A are either 0 or of the form $\pm\sqrt{\mu}$ where μ is a positive eigenvalue of B^TB .