Problem 6

Wednesday, April 15, 2026 9:17 PM

- Denote by $x_1, x_2, \dots, x_{101} \in A$ the elements of A and assume without loss of generality that $x_1 < x_2 < \dots < x_{101}$
- Consider $A_t = \{x+t : x \in A\}$, $t = 1, 2, \dots, 10^6$ the dilations of A by t.

Given $i \in \{0, 1, \dots, 10^6\}$ we need to count the number of A_j with $j > i$ that overlap with A_i.

For A_i, A_j to overlap we must have $\exists m, n \in \{1, 2, \dots, 101\} : x_n \in A_i,\ x_m \in A_j,\ x_n + i = x_m + j$

then $x_n + i = x_m + j \iff x_n - x_m = j - i > 0 \implies x_n > x_m \implies n > m$, hence there are $101 - m$ choices for n.

- We can bound the number of A_j overlapping with A_i by taking into account all possible of m, $1 \le m \le 101$ as follows:

\# of A_j overlapping with A_i $= |\{j \in \{i+1, \dots, 10^6\} : A_i \cap A_j \neq \emptyset\}| \le$

$$\underbrace{|\{j>i : A_i \cap A_j \neq \emptyset, m=1\}|}_{(101-1)} + \underbrace{|\{j>i : A_i \cap A_j \neq \emptyset, m=2\}|}_{(101-2)} + \dots + \underbrace{|\{j>i : A_i \cap A_j \neq \emptyset, m=101\}|}_{(101-101)} = 100 + 99 + \dots + 1 + 0 = 5050$$

- We can now define t_i recursively. Consider the family of sets $S_0 = \{A_1, A_2, \dots, A_{10^6}\}$, $|S_0| = 10^6$

As we showed previously there are at most 5050 sets in S_0 overlapping with $A = A_0$,

let's call them $A_{a_{1,1}}, A_{a_{1,2}}, \dots, A_{a_{1,k_1}}$ where $k_1 \le 5050$

Consider $S_1 = S_0 \setminus \{A_{a_{1,1}}, A_{a_{1,2}}, \dots, A_{a_{1,k_1}}\}$ and choose t_1 to be the minimal index such that $A_{t_1} \in S_1$. Then A, A_{t_1} don't overlap by the choice of $A_{a_{1,i}}$. Also $|S_1| = |S_0| - k_1 \ge 10^6 - 5050$

- For t_2 we can consider $A_{a_{2,1}}, A_{a_{2,2}}, \dots, A_{2,k_2}$, $k_2 \le 5050$ the sets that overlap with A_{t_2}

and let $S_2 = S_1 \setminus (A_{t_1} \cup \{A_{a_{2,1}}, A_{a_{2,2}}, \dots, A_{2,k_2}\})$. Note that A_{t_1} doesn't overlap with the elements of S_2, but also A doesn't overlap with the elements of S_2, because $S_2 \subseteq S_1$.

Also $|S_2| = |S_1| - 1 - k_2 \ge |S_1| - 5051 > 10^6 - 2 \cdot 5051$

- The underlined observation allows us to define $t_3, t_4, \dots, t_{100}$, since $10^6 > 5051 \cdot 100$ ensures that the sets $S_3, S_4, \dots, S_{100}$ are not empty. In fact we can keep going until $\lceil \frac{10^6}{5051} \rceil = 198$