Problem 1

$$1+\sum_{k=0}^{r-1}\binom{r}{k}S_k(n)=1+\sum_{k=0}^{r-1}\binom{r}{k}\sum_{m=1}^{n}m^k=$$

$$=1+\sum_{m=1}^{n}\sum_{k=0}^{r-1}\binom{r}{k}m^k=1+\sum_{m=1}^{n}(1+m)^r-m^r=$$

$$=1+2^r-1^r+3^r-2^r+\dots+(n+1)^r-n^r\ \overset{\text{telescopic}}{\underset{\text{sum}}{=\!=}}\ 1-1^r+(n+1)^r\Longrightarrow$$

$$\Longrightarrow 1+\sum_{k=0}^{r-1}\binom{r}{k}S_k(n)=(n+1)^r$$

Problem 2

$$\binom{n}{k}=\frac{(n-1)!}{k!}\left(\frac{n-k+k}{(n-k)!}\right)=\frac{(n-1)!}{k!}\left(\frac{1}{(n-k-1)!}+\frac{k}{(n-k)!}\right)\Longrightarrow$$

$$\Longrightarrow\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}\quad(\text{Pascal's idenzizy})$$

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k}=1+\sum_{k=1}^{m}(-1)^k\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)=$$

$$1-\binom{n-1}{1}-\binom{n-1}{0}+\binom{n-1}{2}+\binom{n-1}{1}-\dots+$$

$$+(-1)^m\binom{n-1}{m}+(-1)^m\binom{n-1}{m-1}=1-\binom{n-1}{0}+(-1)^m\binom{n-1}{m}\Longrightarrow$$

$$\Longrightarrow\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^m\binom{n-1}{m}$$

Problem 3

We observe from $(\sqrt{m}+\sqrt{m-1})^2 = 2m-1+2\sqrt{m(m-1)}$ that

$(\sqrt{m}+\sqrt{m-1})^{2n} = A_n + B_n\sqrt{m(m-1)}$, where A_n, B_n integers

This holds as: $((\sqrt{m}+\sqrt{m-1})^2)^n =$

$\underbrace{(A_1+B_1\sqrt{m(m-1)})\cdot \ldots \cdot (A_1+B_1\sqrt{m(m-1)})}_{n \text{ times}}$, so if we compute

the product, we get a sum consisting

of terms of the form $A_1^{\,i}(B_1\sqrt{m(m-1)})^j$.

Hence: $(\sqrt{m}+\sqrt{m-1})^{2n} = (\sqrt{p}+\sqrt{p-1})^2 \Leftarrow$

$\Rightarrow A_n + B_n\sqrt{m(m-1)} = 2p-1+2\sqrt{p(p-1)}$

$A_n = 2p-1 \Rightarrow p = \dfrac{A_n+1}{2}$,

If A_n is odd then p is an integer

Induction: For $n=1$: $A_1 = 2m-1 \Rightarrow A_1$ odd

Assume it holds for $n=k$

for $n=k+1$: $(A_k+B_k\sqrt{m(m-1)})(A_1+B_1\sqrt{m(m-1)}) =$

$= \overbrace{A_k A_1}^{\text{odd}} + \overbrace{B_k B_1 (m(m-1))}^{\text{even}} + \ldots \Rightarrow A_{k+1}$ odd

So such a number p exists.

Problem 4

Denote by A_n the set of positive integers smaller than 1000 that are divided by n. Then denote by S the set of positive integers that are divided by 2, 3 or 5. By the inclusion-exclusion principle:

$$|S| = |A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| -$$
$$- |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5| = \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor -$$
$$- |A_{2\cdot 3}| - |A_{2\cdot 5}| - |A_{3\cdot 5}| + |A_{2\cdot 3\cdot 5}| = 500 + 333 + 200 - \left\lfloor \frac{1000}{6} \right\rfloor -$$
$$- \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor = 1033 - 166 - 100 - 66 + 33 =$$
$$= 734$$

So for the set S' of the numbers that aren't divisible by 2, 3, 5: $|S'| = 1000 - |S| \Rightarrow |S'| = 266$

Problem 5

For every single word we can use the filter

$f_w = \frac{1 + (-1)^{\#a}}{2} \cdot \frac{1 + (-1)^{\#b}}{2}$, where $\#a$ is the number of a's and $\#b$ number of b's. (If a, b even $f = 1$, else $f = 0$

$$\sum_{\text{all word}} f_w = \sum \frac{1}{4} + \frac{(-1)^{\#a}(-1)^{\#b}}{4} + \frac{(-1)^{\#a} + (-1)^{\#b}}{4} = \frac{1}{4}\left(\sum 1 + \sum (-1)^{\#a+\#b} + 2\sum (-1)^{\#a}\right)$$

Because of symmetry $\sum (-1)^{\#a} = \sum (-1)^{\#b}$

In addition the number of total possible words consisting of n letters is $4^n \Rightarrow \sum_{\text{all words}} 1 = 4^n$

For even a, with $\#a = 2k$ we have 3 choices for other letter, so we add up $\binom{n}{2k} 3^{n-2k}$ for every a:

$$S_{even} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 3^{n-2k} = \sum_{k \text{ even}}^{n - n \bmod 2} \binom{n}{k} 3^{n-k} = \frac{1}{2}\left(\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-1)^k + \right.$$

$$\left. + \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-1)^k \right) = \frac{(3-1)^n + (3+1)^n}{2} = \frac{4^n + 2^n}{2}$$

For odd a: $S_{odd} = 4^n - S_{even} = \frac{4^n - 2^n}{2}$

Hence, $\sum_{\text{all words}} (-1)^{\#a} = S_{even} - S_{odd} = 2^n$

Next there is a state of (a,b) for every word:

Denote E = even, O = odd, the four states a word can be in is (E,E), (O,O), (E,O), (O,E). Now denote by S_{xy} the set of words in state (x,y).

Then $\sum_{\text{all words}} (-1)^{\#a + \#b} = |S_{ee}| + |S_{oo}| - |S_{eo}| - |S_{eo}|$

We've got
- $|S_{eo}| = |S_{oe}|$ because of symmetry
- $|S_{ee}| + |S_{oe}| = \frac{4^n + 2^n}{2}$, $|S_{oo}| + |S_{eo}| = \frac{4^n - 2^n}{2}$

But $\cancel{|S_{ee}|} = \sum_{\text{all words}} f_w = 4^{n-1} + 2^{n-1} + \cancel{|S_{ee}|} + |S_{oo}| - 2|S_{oe}| \Rightarrow$

$|S_{oo}| = 2|S_{oe}| - (4^{n-1} + 2^{n-1}) \Longrightarrow 3|S_{oe}| = \frac{4^n}{2} - \frac{2^n}{2} + \frac{4^n}{4} + \frac{2^n}{2} \Longrightarrow$

$\Longrightarrow |S_{oe}| = 4^{n-1} \Longrightarrow |S_{ee}| = \frac{4^n}{2} - \frac{4^n}{4} + \frac{2^n}{2} \Longrightarrow$

$\Longrightarrow |S_{ee}| = 4^{n-1} + 2^{n-1}$

Problem 7

$$\sum_{k=1}^{n} \binom{n}{k}^2 = \sum n\binom{n-1}{k-1}\binom{n}{k} = n\sum\binom{n-1}{k-1}\binom{n}{k}$$

We take the generating functions $G_1(x) = \sum_{i=0}^{n}\binom{n}{i}x^i =$

$= (1+x)^n$, $G_2(x) = \sum_{j=0}^{n-1}\binom{n-1}{j}x^j = (1+x)^{n-1}$

$$G_1(x)\,G_2(x) = (1+x)^n(1+x)^{n-1} \Rightarrow \sum_{i=0}^{n}\binom{n}{i}x^i\sum_{j=0}^{n-1}\binom{n-1}{j}x^j =$$

$$= \sum_{k=0}^{2n-1}\binom{2n-1}{n}x^k$$

for the coefficient of x^n:

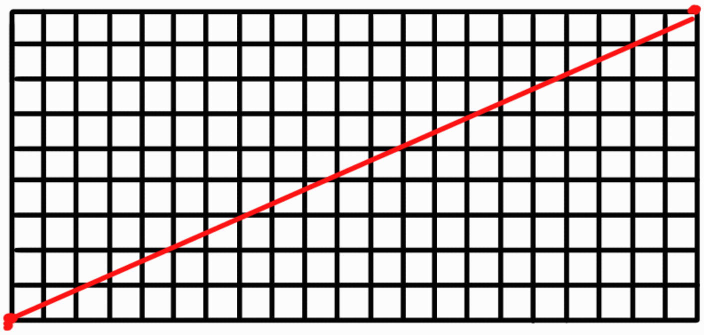
$$\binom{2n-1}{n} = \binom{n}{1}\binom{n-1}{n-1} + \binom{n}{2}\binom{n-1}{n-2} + \dots + \binom{n}{n}\binom{n-1}{0} =$$

$$= \sum_{k=1}^{n}\binom{n}{k}\binom{n-1}{n-k} = \sum_{k=1}^{n}\binom{n}{k}\binom{n-1}{k-1} \xRightarrow{\binom{2n-1}{n} = \binom{2n-1}{n-1}}$$

$$\Rightarrow n\sum_{k=1}^{n}\binom{n}{k}\binom{n-1}{k-1} = n\binom{2n-1}{n-1} \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

Problem 8



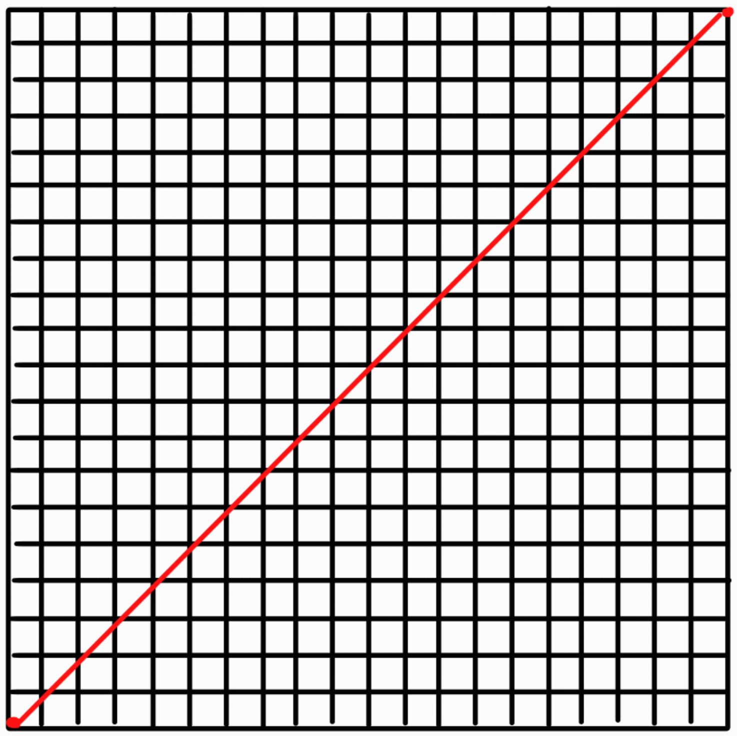
324

we observe that the number of cubes that the diagonal passes through are equal to the sum of the dimensions of the solid, minus the ones we count 2 times.

150

So we can use the Inclusion - Exclusion principle

324



375

Denote by A_j, the set of j cubes that belong to a side of the solid. $A_j \cap A_i$ counts the amount of common cubes the diagonal passes through.

$|A_i \cap A_j| = \gcd(i, j)$. (The gcd counts the intersection points, so no cube is cut in fractions. That means that means that the point $\left(\frac{a}{d}, \frac{b}{d}\right)$ of the diagonal has integer coordinates, hence $d \mid a$ and $d \mid b$.)

$|A_{150} \cup A_{324} \cup A_{375}| = |A_{150}| + |A_{324}| + |A_{375}| -$
$|A_{150} \cap A_{324}| - |A_{150} \cap A_{375}| - |A_{324} \cap A_{375}| +$
$+ |A_{150} \cap A_{375} \cap A_{324}| = 150 + 324 + 375 - 6 - 75 - 3 + 3 = 768$

Problem 9

If n players play against others then each player will play $n-1$ games so for player i $x_i + y_i = n-1$

Also there will be the same amount of wins and losses,

so $x_1 + x_2 + \dots + x_n = y_1 + y_2 + \dots + y_n = \frac{n(n-1)}{2}$

$x_1 - y_1 + x_2 - y_2 + \dots + x_n - y_n = 0.$

Now we multiply everything by $\prod_{i=1}^{n} x_i + y_i$

$(x_1^2 - y_1^2)(\prod_{i \neq 1}(x_i + y_i)) + (x_2^2 - y_2^2)\prod_{i \neq 2}(x_i + y_i) + \dots +$

$+ (x_n^2 - y_n^2)(\prod_{i \neq n} x_i + y_i) = 0 \Longrightarrow (x_1^2 - y_1^2)(n-1)^{n-1} +$

$+ (x_2^2 - y_2^2)(n-1)^{n-1} + \dots + (x_n^2 - y_n^2)(n-1)^{n-1} = 0 \Rightarrow$

$\Longrightarrow x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2$

Problem 10

Consider the grid $1 \le x \le p'$, $1 \le y \le q'$

Then $S_1 = \sum_{k=1}^{p'} \left\lfloor x \frac{q}{p} \right\rfloor$ counts the points

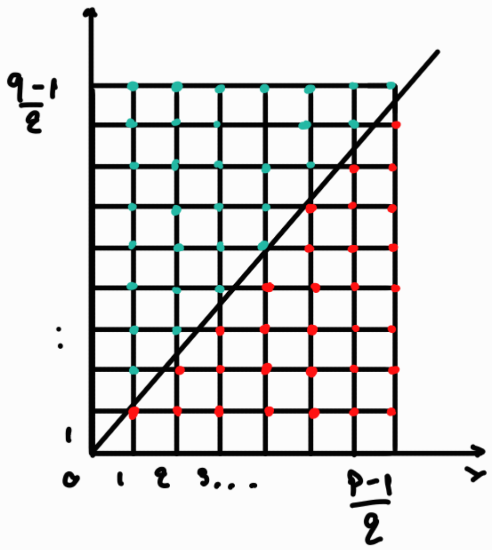
(x, y) with $1 \le x \le p'$ and $1 \le y \le \left\lfloor \frac{xq}{p} \right\rfloor$ (This comes from the definition of floor: $\lfloor x \rfloor = \#\{y \in \mathbb{Z} : 1 \le y \le x\}$.

so the points under the line $y = \frac{q}{p} x$

$S_2 = \sum_{k=1}^{q'} \left\lfloor y \frac{p}{q} \right\rfloor$ counts the points (x, y) with $1 \le y \le q'$,

$1 \le x \le \left\lfloor y \frac{p}{q} \right\rfloor$, so the points on the left of the line $x = \frac{p}{q} y \Rightarrow$

$\Rightarrow$ the points over $y = \frac{q}{p} x$.



Hence if we assign the top right point of each square to it, the total amount of points equals the number of squares of the grid

This is equivalent to a rectangle of sides $\frac{q-1}{2}$, $\frac{p-1}{2}$ so $S_1 + S_2 = \frac{q-1}{2} \cdot \frac{p-1}{2}$