Math Competition Preparation Seminar

Combinatorics

March 23, 2026

1 Theory/Background

1.1 Binomial Coefficient-Expansion

The binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ counts the number of ways one can choose k objects from given n. Binomial coefficients show up in Newton's binomial expansion:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1.2 Inclusion-Exclusion Principle (Boole-Sylvester Formula)

Let $A_1, \ldots, A_n$ be a family of finite sets. Then the number of elements in the union $A_1 \cup \cdots \cup A_n$ is given by

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|.$$

2 Problems

Problem 1. Let $S_k(n) = 1^k + 2^k + \cdots + n^k$, where k is a non-negative integer. Show that

$$1 + \sum_{k=0}^{r-1} \binom{r}{k} S_k(n) = (n+1)^r.$$

Problem 2. Compute the sum:

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k}.$$

Problem 3. Prove that for natural numbers m and n, there exists a natural number p such that the identity

$$\left(\sqrt{m} + \sqrt{m-1}\right)^n = \sqrt{p} + \sqrt{p-1}$$

holds.

Problem 4. How many integers less than 1000 are not divisible by 2, 3, or 5?

Problem 5. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

- w consists of n letters, all of them are from the alphabet $\{a, b, c, d\}$.
- w contains an even number of letters a.
- w contains an even number of letters b.

Problem 6. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100,$$

are pairwise disjoint.

Problem 7. Prove the combinatorial identity

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

Problem 8. A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?

Problem 9. A number n of tennis players take part in a tournament in which each of them plays exactly one game with each of the others. If x_i and y_i denote the number of victories, respectively losses, of the ith player, $i = 1, 2, \ldots, n$, show that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2.$$

Problem 10. Let p and q be odd, coprime positive integers. Set $p' = \frac{p-1}{2}$ and $q' = \frac{q-1}{2}$. Prove the identity

$$\left(\left\lfloor \frac{q}{p} \right\rfloor + \left\lfloor \frac{2q}{p} \right\rfloor + \cdots + \left\lfloor \frac{p'q}{p} \right\rfloor \right) + \left(\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \cdots + \left\lfloor \frac{q'p}{q} \right\rfloor \right) = p'q'.$$