Problem 1

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i) First notice that $a_n > 1 \Rightarrow \frac{a_n}{a_n+1} > \frac{1}{2}$ and $a_n \leq 1 \Rightarrow \frac{a_n}{a_n+1} > \frac{a_n}{2}$

$$\sum_{n=1}^{N} \frac{a_n}{1+a_n} = \sum_{\substack{1 \leq n \leq N \\ a_n > 1}} \frac{a_n}{1+a_n} + \sum_{\substack{1 \leq n \leq N \\ a_n \leq 1}} \frac{a_n}{1+a_n} > \frac{1}{2} \sum_{\substack{1 \leq n \leq N \\ a_n > 1}} 1 + \frac{1}{2} \sum_{\substack{1 \leq n \leq N \\ a_n \leq 1}} a_n$$

- If $|\{n \in \mathbb{N}^*: a_n > 1\}| = \infty$ then $\sum_{\substack{1 \leq n \leq N \\ a_n > 1}} 1 \xrightarrow{N \to \infty} \infty$

- If $|\{n \in \mathbb{N}^*: a_n > 1\}| = n_0$ then $\sum_{\substack{n \geq 1 \\ a_n < 1}} a_n = \infty$ since $\sum_{n=1}^{\infty} a_n = \infty$

and we ignore a finite number of terms. Therefore $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} = \infty$

ii) Since $a_n > 0$, s_n is increasing, so $s_{N+i} < s_{N+k}$, $i \in \{1, 2, .., k-1\}$

$$\frac{a_{N+1}}{s_{N+1}} + .. + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + .. + \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1} + .. + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_{N+1}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

iii) $$\frac{a_n}{s_n^2} \geq \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

Problem 3

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Our strategy is to partition the natural numbers based on the number of digits.

For a specific number of digits d, there are $8 \cdot 9^{d-1}$ numbers with d digits not containing 9 in their decimal represantation.

That's because there are 8 possible choices for the first digits, $1, 2, .., 8$ and 9 possible choices for the rest $d-1$ digits.

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{n} = \sum_{d=1}^{\infty} \sum_{10^{d-1} \leq n < 10^d} \frac{\varepsilon_n}{n} \leq \sum_{d=1}^{\infty} \sum_{10^{d-1} \leq n < 10^d} \frac{\varepsilon_n}{10^{d-1}} = \sum_{d=1}^{\infty} \frac{1}{10^{d-1}} \sum_{10^{d-1} \leq n < 10^d} \varepsilon_n = \sum_{d=1}^{\infty} \frac{8 \cdot 9^{d-1}}{10^{d-1}} = 8 \sum_{d=1}^{\infty} \left(\frac{9}{10}\right)^{d-1} < \infty$$

Problem 5

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$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} > \sum_{p:\text{prime}} \frac{\varphi(p)}{p^2} = \sum_{p:\text{prime}} \frac{p-1}{p^2} = \sum_{p:\text{prime}} \frac{1}{p} - \sum_{p:\text{prime}} \frac{1}{p^2} = \infty \quad \text{because} \sum_{p:\text{prime}} \frac{1}{p} = \infty$$

Problem 6

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- Take $N \in \mathbb{N}^*$. $\sum_{1\le n,m\le N} \frac{1}{n^2+m^2} > \sum_{1\le n,m\le N} \frac{1}{(n+m)^2} > \sum_{\substack{1\le n,m\le N\\ n+m\le N}} \frac{1}{(n+m)^2} = \sum_{s=2}^{N} \sum_{\substack{1\le n,m\le N\\ n+m=s}} \frac{1}{(n+m)^2} =$

$\sum_{s=2}^{N} \frac{1}{s^2} \sum_{\substack{1\le n,m\le N\\ n+m=s}} 1 = \sum_{s=2}^{N} \frac{1}{s^2}(s-1) = \sum_{s=2}^{N} \frac{1}{s} - \sum_{s=2}^{N} \frac{1}{s^2}$

where we used the fact that s can be expressed as $s=n+m$ in $s-1$ ways. There are $s-1$ choices for n, $1\le n\le s-1$ each corresponding to a unique value of m. Therefore $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \Rightarrow \sum_{n,m\ge 1} \frac{1}{n^2+m^2} = \infty$

- A less innovative idea would be to use Fourier series

Consider $f(t) = e^{mt}$, $t\in[-\pi,\pi)$, $m\in\mathbb{N}^*$

$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{mt-int}\,dt = \frac{1}{2\pi}\frac{1}{m-in} e^{(m-in)t}\Big|_{-\pi}^{\pi} = \frac{\sinh(m+in)\pi}{\pi}\frac{1}{m-in}$

- f is differentiable at $t=0$ hence its Fourier series evaluated at $t=0$ converges to $f(0)$

- $f(0) = \sum_{n=-\infty}^{\infty} a_n e^{int}\Big|_{t=0} \Rightarrow 1 = \sum_{n=-\infty}^{\infty} a_n \Rightarrow 1 = a_0 + \sum_{n=1}^{\infty}(a_n + a_{-n}) \Rightarrow$

$\sum_{n=1}^{\infty}\left(\frac{\sinh(m+in)\pi}{\pi}\frac{1}{m-in} + \frac{\sinh(m-in)\pi}{\pi}\frac{1}{m+in}\right) = 1 - \frac{\sinh m\pi}{\pi m} \Rightarrow$

Problem 10

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Take $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} a_n^2$, $\sum_{n=1}^{\infty} b_n^2$ converge $\exists N_1, N_2 \in \mathbb{N}^*$: $\sum_{n=N_1}^{\infty} a_n^2 < \varepsilon$, $\sum_{n=N_2}^{\infty} b_n^2 < \varepsilon$

Let $M \geq N \geq \max\{N_1, N_2\}$, then

Cauchy-Schwarz: $\left(\sum_{n=N}^{M} a_n b_n\right)^2 \leq \sum_{n=N}^{M} a_n^2 \cdot \sum_{n=N}^{M} b_n^2 \leq \sum_{n=N_1}^{\infty} a_n^2 \cdot \sum_{n=N_2}^{\infty} b_n^2 < \varepsilon \cdot \varepsilon = \varepsilon^2 \Rightarrow$

$$\left|\sum_{n=N}^{M} a_n b_n\right| < \varepsilon$$

Thus $\forall \varepsilon > 0$: $\exists n_0 (= \max\{N_1, N_2\}) \in \mathbb{N}^*$: $\left|\sum_{n=N}^{M} a_n b_n\right| < \varepsilon$ $\forall M, N > n_0$

so $\sum_{n=1}^{N} a_n b_n$: Cauchy $\Rightarrow \sum_{n=1}^{\infty} a_n b_n$: converges.

Notice that since a_n, b_n are not necessarily positive it is not sufficient to prove that $\left|\sum_{n=1}^{N} a_n b_n\right| \leq \sqrt{\sum_{n=1}^{\infty} a_n^2 \cdot \sum_{n=1}^{\infty} b_n^2}$ using an argument similar to the above.