Math Competition Preparation Seminar

Series Convergence

March 6, 2026

1 Theory/Background

1.1 Main Ideas

- Telscopic sums: In order to simplify a sum express each addend as the difference of two consecutive terms of some sequence and simplify all intermediate terms. For example, if $a_n = b_n - b_{n+1}$ then

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} b_n - \sum_{n=1}^{N} b_{n+1} = b_1 - b_{N+1}$$

1.2 Cauchy-Schwarz Inequality

Let $\{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N \subset \mathbb{R}, n \ \in \mathbb{N}$. Then

$$\sum_{n=1}^{N} a_n^2 \sum_{n=1}^{N} b_n^2 \geq \left(\sum_{n=1}^{N} a_n b_n \right)^2$$

1.3 Stirling's formula

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

2 Problems

Problem 1 Let $\{a_n\}_{n\geq 1} \subset \mathbb{R}^+$, $s_n = a_1 + \cdots + a_n$. If $\sum_{n=1}^{\infty} a_n$ diverges. Prove the following:

- i) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges.
- ii) $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$ and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ also diverges.
- iii) $\frac{a_n}{{s_n}^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$ and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ converges.

Problem 2 Let $\{a_n\}_{n\geq 1} \subset \mathbb{R}^+$, $s_n = a_1 + \cdots + a_n$. If $\sum_{n=1}^{\infty} a_n$ diverges. Show that the following series converges $\forall \epsilon > 0$:

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n^{1+\epsilon}}$$

Problem 3 Let ϵ_n, such that ϵ_n whenever the decimal representation of n contains the digit 9 and $\epsilon_n = 1$ otherwise. Determine whether the following series converges

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{n}$$

Problem 4 Determine whether $\sum_{n=1}^{\infty} \frac{sin(n^2)}{n}$ converges.

Problem 5 Determine whether $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^2}$ converges. Here $\phi(n)$ denotes Euler's totient function evaluated at $n \in \mathbb{N}$.

Problem 6 Determine whether $\sum_{n,m\geq 1} \frac{1}{n^2+m^2}$ converges.

Problem 7 Let $\{a_n\}_{n\geq 1} \subset \mathbb{R}^+$, such that $\lim_{n\to\infty} a_n = 0$. Prove the following:

- i) If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} min(a_n, \frac{1}{n})$.
- ii) If $\sum_{n=1}^{\infty} \frac{a_n}{n}$ diverges, then so does $\sum_{n=1}^{\infty} min(a_n, \frac{1}{lnn})$.

Problem 8 Let $\{a_n\}_{n\geq 1} \subset \mathbb{R}^+$, such that $\sum_{n=1}^{\infty} a_n$ converges. Show that the following series also converges

$$\sum_{n=1}^{\infty} \frac{a_n}{ln(n+1)} ln(a_n)$$

Problem 9 Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges, where

$$a_n = \left(\frac{(2n-1)!!}{(2n)!!}\right)^2, \; n \in \mathbb{N}$$

Problem 10 Let $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1} \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} a_n^2, \sum_{n=1}^{\infty} b_n^2$ both converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ also converges.