Problem 4

Sunday, March 1, 2026 5:26 PM

- Our idea is to consider all possible rotations τ. As Y is rotated, every point $y \in Y$ meets each point $x \in X$ exactly once. Thus each $x \in X$ is met exactly $|Y|$ times. Let's be rigorous:
- Let Y_i be the clockwise rotation of Y by $\frac{2\pi}{n} i$, $i = 0, 1, \dots, n-1$
- Fix $x_0 \in X$. For each $y \in Y$ there is a unique rotation of Y, that sends y on top of x_0. Thus $\chi_{\tau(Y)}$, the characteristic function of $\tau(Y)$, becomes 1 exactly $|Y|$ times, once for each $y \in Y$
- $\sum_{\tau:\text{rot}} |X \cap \tau(Y)| = \sum_{\tau:\text{rot}} \sum_{x \in X} \chi_{\tau(Y)}(x) = \sum_{x \in X} \sum_{\tau:\text{rot}} \chi_{\tau(Y)}(x) = \sum_{x \in X} |Y| = |X||Y|$
- So if τ_0 is the rotation that maximises $|X \cap \tau(Y)|$, then

$n|X \cap \tau_0(Y)| = |X \cap \tau_0(Y)| \cdot \sum_{\tau:\text{rot}} 1 = \sum_{\tau:\text{rot}} |X \cap \tau_0(Y)| \geq \sum_{\tau:\text{rot}} |X \cap \tau(Y)| = |X||Y| \Rightarrow$

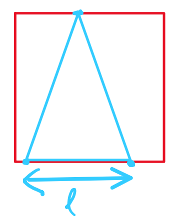
$|X \cap \tau_0(Y)| \geq \frac{|X||Y|}{n}$

Problem 5

Sunday, March 1, 2026 5:27 PM

- We first consider triangles with all 3 of their vertices lying on the sides of the square. We distinguish 2 cases:

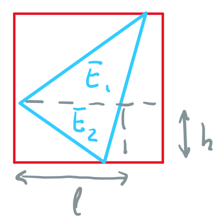
- <u>Case 1:</u> Assume two vertices lie on the same side



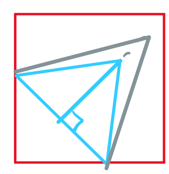
Then the triangle's area is $E = \frac{\ell h}{2} = \frac{\ell}{2}$

But $\ell \leq 1$, hence $E = \frac{\ell}{2} \leq \frac{1}{2}$, equality is obtained when $\ell = 1$, i.e. the triangle and the square have a common side.

- <u>Case 2:</u> Assume the 3 vertices lie on different sides of the square, as the figure suggests:



$$E = E_1 + E_2 = \frac{(1-h)\ell}{2} + \frac{h\ell}{2} = \frac{\ell}{2} \leq \frac{1}{2}$$



- Finally whenever at least one vertex of the triangle doesn't lie on one of the square's sides we can increase its height, getting a new triangle inside the square with larger area. Hence such a triangle can have maximal area.

The answer is indeed $\frac{1}{2}$, since we have shown it is attainable

Problem 9

Sunday, March 1, 2026 5:50 PM

- Associate +1 with each '+' sign and -1 with each '-' sign
We will show that the product of these '+1's, '-1's remains invariant. In each operation there are 3 possible outcomes (denote by p the product before applying the operation):

- We may remove +, + and replace them by a +.
Then removing +1 two times doesn't change p, and multiplying by +1, also leaves p unchanged.
- We may remove +, − and replace them by a −.
Then removing +1, −1 changes p to $p' = -p$, but then multiplying by −1, changes p' to $p'' = -p' = p$
- We may remove −, − and replace them by a +.
Then removing −1 two times changes p to $p' = \frac{p}{(-1)^2} = p$
multiplying by +1, also leaves p unchanged.

We conclude that the product p remains constant, no matter what operation we apply.
After multiple operations we end up with only one sign which is determined only by p ('−' if $p = -1$, '+' otherwise)

Problem 10

Sunday, March 1, 2026 5:28 PM

- We will assume on the contrary that no car can complete the lap by reaching all the next cars and collecting their gas.
- Now consider the longest possible distance each car could cover by using all its fuel and all the fuel it can collect.

  Think of two consecutive cars, if the one at the back can reach the one in front it will cover a longer distance, because it will reach it with remaining gas ≥ 0.
- Let c_0 be the car that can cover the maximal possible distance. According to the previous observation, no car behind it can reach it, because it would cover an even longer distance which is impossible.
- Therefore there are cars that cannot be reach by cars behind them. Let all these cars use all the fuel they can. Then all the fuel will be used (because otherwise one car that can't be reached wouldn't have used all its fuel). This implies that all the fuel will be used without all parts of the track being covered by some car, a contradiction.

Problem 14

Sunday, March 1, 2026 5:28 PM

- Define $a_i = \#$ chess es played until the i-th day
- Observe that $a_j - a_i$ is equal to the number of games played between days $i+1$ and j.

  We need to show that $\exists i,j : i<j,\ a_j - a_i = 20 \Leftrightarrow a_j = a_i + 20$
- Also, since the chess player plays at least 1 game every day, a_i is increasing.
- Let $m \in \mathbb{N}^*$. Consider the following $14m$ numbers

  $a_1, a_2, \dots, a_{7m}, a_1+20, a_2+20, \dots, a_{7m}+20$

  Then since a_i : increasing $a_i \le a_{7m}$, $i \in \{1, 2, \dots, 7m\}$
- The chess player plays at most 12 games per week, thus in m weeks he plays $a_{7m} \le 12m$ games, so

$1 \le a_i \le a_{7m} \le 12m$, $a_i + 20 \le 12m + 20$

- All numbers are bounded by $12m+20$, hence choosing m large enough (say $m=11$) such that $12m+20 < 14m$ and applying the pigeonhole principle

  (pigeons: $a_1, \dots, a_{7m}, a_1+20, \dots, a_{7m}+20$,

  nests: integers from 1 to $12m+20$)

  implies that some two of these numbers are equal.
- However $i \ne j \Rightarrow a_i \ne a_j$ because a_i is strictly increasing and hence $i \ne j \Rightarrow a_i + 20 \ne a_j + 20$

- We conclude that $\exists i, j \in \{1, \dots, 7m\}: a_j = a_i + 20 \Leftrightarrow a_j - a_i = 20$ hence the player played exactly 20 games during the consecutive days $i+1, i+2, \dots, j$.