# Math Competition Preparation Seminar

Discrete

February 27, 2026

## 1 Theory/Background

### 1.1 Pigeonhole principle

If $kn+1$ objects, $k \geq 1$, are distributed among $n$ boxes, one of the boxes will contain at least $k+1$ objects.

### 1.2

Main Ideas

- Invariants(revisited): A number or a property that characterizes a mathematical object. Very useful when dealing with repeatedly applied transformations. Induction is sometimes useful.
- Extremal Principle: Consider an object of a set that maximizes some function.

## 2 Problems

**Problem 1** Let $n \in \mathbb{N}^*$.
(i) Suppose that $0 < x_1 < \cdots < x_N < 2n+1$ are such that $|kx_i - x_j| \geq 1$ for all natural numbers $i, j, k : 1 \leq i < j \leq N$. At most how large is $N$?
(ii) At most how many real numbers can be chosen from the open interval $(0, (3n+1)/2)$ if none is at distance less than 1 from an odd multiple of another?

**Problem 2** Is there a convex polyhedron which contains a point whose perpendicular projection on the plane of every face is outside the face? What if we want the projection to be in the interior of the face?

**Problem 3** Show that among all polygons with $N$ vertices and fixed perimeter, the regular N-gon has the largest area.

**Problem 4** Let $X$ and $Y$ be subsets of the vertex set of a regular n-gon. Show that there is a rotation $\tau$ of this polygon such that $|X \cap \tau(Y)| \geq \frac{|X||Y|}{n}$ where, as usual, $|Z|$ denotes the number of elements in a finite set Z.

**Problem 5** Find the maximal area of a triangle inside a square of side length 1.

**Problem 6** Show that every $n \times n$ matrix whose entries are $1, 2, ..., n^2$ in some order has two neighbouring entries (in a row or in a column) that differ by at least $n$.

**Problem 7** In how many ways can one arrange $k$ rooks on a chessboard with $m$ rows and $n$ columns so that no rook can attack another?

**Problem 8** Does the sequence of squares contain an infinite arithmetic subsequence?

**Problem 9** There are several signs $+$ and $-$ on a blackboard. You may erase two signs and write, instead, $+$ if they are equal and $-$ if they are unequal. Then, the last sign on the board does not depend on the order of erasure.

**Problem 10** There are $n$ identical cars on a circular track. Together they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around.

**Problem 11** Let $M$ be the largest distance among six distinct points of the plane, and let $m$ be the smallest of their mutual distances. Prove that $\frac{M}{m} \geq \sqrt{3}$.

**Problem 12** The integers $1, ..., n$ are arranged in order. In one step you may take any four integers and interchange the first with the fourth and the second with the third. Prove that, if $\frac{n(n-1)}{2}$ is even, then by means of such steps you may reach the arrangement $n, n-1, ..., 1$. But if $\frac{n(n-1)}{2}$ is odd, you cannot reach this arrangement.

**Problem 13** Let $x_1 = x_2 = x_3 = 1$ and $x_{n+3} = x_n + x_{n+1}x_{n+2}$ for all positive integers $n$. Prove that for any positive integer m there is an index $k$ such that m divides $x_k$.

**Problem 14** A chess player trains by playing at least one game per day, but, to avoid exhaustion, no more than 12 games a week. Prove that there is a group of consecutive days in which he plays exactly 20 games.

**Problem 15** Prove that among any eight positive integers less than 2004 there are four, say $a, b, c, d$, such that $4 + d \leq a + b + c \leq 4d$.