Math Competition Preparation Seminar

Linear Algebra

January 9, 2026

1 Theory/Background

1.1 Defintion(Spectrum)

The spectrum $\sigma(A)$ of a square matrix A is the multiset (i.e. a set which may contain an element multiple times) of its eigenvalues.

1.2 Theorem

Let $n \in \mathbb{N}^*$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then $\exists P, J, Q \in \mathbb{M}_n(\mathbb{C})$, where P, Q are invertible, such that

$$A = PJQ,\ J = \begin{bmatrix} I_r & O_{r\times(n-r)} \\ O_{(n-r)\times r} & O_{(n-r)\times(n-r)} \end{bmatrix}$$

where $r = rank(A)$. This decomposition of A is derived by applying Gaussian elimination on A.

1.3 Theorem

Let $n \in \mathbb{N}^*$ and $A, B \in \mathbb{M}_n(\mathbb{C})$. Then $\sigma(AB) = \sigma(BA)$.

1.4 Spectral Mapping Theorem

Let $A \in \mathbb{M}_n(\mathbb{C})$, $\lambda_1, \lambda_2, .., \lambda_n$ the eigenvalues of A(not necessarily distinct) and P a polynomial. Then the eigenvalues of $P(A)$ are $P(\lambda_1), P(\lambda_2), .., P(\lambda_n)$.

2 Problems

Problem 1 Let $A, B \in \mathbb{M}_n(\mathbb{C}) : AB = BA$. Show that if $\det(A + B) \geq 0$ then $\det(A^k + B^k) \geq 0$.

Problem 2 Let $m, n \in \mathbb{N}^*$ and $A \in \mathbb{M}_{m\times n}(\mathbb{C}), B \in \mathbb{M}_{n\times m}(\mathbb{C})$. Prove that

$$\begin{vmatrix} I_n & B \\ A & I_m \end{vmatrix} = \begin{vmatrix} I_m & A \\ B & I_n \end{vmatrix}$$

Problem 3 Let $n \in \mathbb{N}^*$ and $A, B \in \mathbb{M}_n(\mathbb{C})$. Show that $\det(I_n + AB) = \det(I_n + BA)$.

Problem 4 Let $n \in \mathbb{N}^*$. Consider $2n+1$ real numbers such that when discarding any of them you can split the remaining $2n$ into two sets each consisting of n numbers with equal sums. Show that all the numbers are equal.

Problem 5 Let $n \in \mathbb{N}^*$. Consider the set S consisting of $2n-1$ different irrational numbers. Prove that $\exists x_1, x_2, .., x_n \in S$ such that for all $a_1, a_2, .., a_n \in \mathbb{Q}_{\geq 0}$ not all zero, $a_1x_1 + a_2x_2 + .. + a_nx_n \notin \mathbb{Q}$.

Problem 6 Let $n \in \mathbb{N}^*$ and $A \in \mathbb{M}_n(\mathbb{C})$. Prove that $\exists B \in \mathbb{M}_n(\mathbb{C})$ such that $ABA = A$.

Problem 7 Let $A \in \mathbb{M}_n(\mathbb{C})$ be invertible. Show that

$$rank \begin{pmatrix} A & B \\ C & D \end{pmatrix} = rank(A) + rank(D - CA^{-1}B)$$

Problem 8 Assuming matrices A, B, X, Y have appropriate dimensions, prove that if $rank(AB) = rank(B)$ then

$$ABX = ABY \iff BX = BY$$

Problem 9 Compute the following determinant

$$\begin{vmatrix} 1 + x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & 1 + x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & 1 + x_ny_n \end{vmatrix}$$