Problem 7(Day 1) (Foivos)

Denote by $s(t)$ the distance covered by the runner in km after t minutes.

Then $s:[0,30]\to[0,6]$ and $s(0)=0$, $s(30)=6$

Assume on the contrary that the runner always covered a distance larger than 1km in 5 minutes. This is interpreted as $s(t+5)-s(t)>1$ $\forall t\in[0,25]$

Then $6=s(30)-s(0)=(s(30)-s(25))+(s(25)-s(20))+..+(s(5)-s(0))>1+1+..+1=6$ contradiction

hence $\exists t_1\in[0,25]: s(t_1+5)-s(t_1)\leq 1$

A similar argument shows that $\exists t_2\in[0,25]: s(t_2+5)-s(t_2)\geq 1$

Consider $g(t)=s(t+5)-s(t)$, $t\in[0,25]$, then assuming s is continuous, g is also continuous

and $g(t_1)\geq 1$, $g(t_2)\leq 1$ hence by the intermediate value theorem

$\exists t_0\in(\min\{t_1,t_2\},\max\{t_1,t_2\}): g(t_0)=1$ or $g(t_1)=1$ or $g(t_2)=1$

thus $\exists t_0\in[\min\{t_1,t_2\},\max\{t_1,t_2\}]:$ $s(t_0+5)-s(t_0)=1$

Problem 13(Day 1)

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Assume on the contrary $f(x)>1 \ \forall x\in\mathbb{R}$

Then $f''(x)=e-e^{f(x)}<0$, so f is concave, hence not constant, $\exists x_0\in\mathbb{R}: f'(x_0)\neq 0$

Case 1: $f'(x_0)>0$ Since f is concave it lies under its tangents

$f(x)\leq f'(x_0)(x-x_0)+f(x_0) \xrightarrow[f'(x_0)>0]{x\to-\infty} -\infty$ contradiction since $f(x)>1$

Case 2: $f'(x_0)<0$ Similarly $f(x)\leq f'(x_0)(x-x_0)+f(x_0)\xrightarrow{x\to+\infty}-\infty$

Problem 1(Day 2)

$$a_n < a_{n-1} + \frac{1}{n^2} < a_{n-2} + \frac{1}{(n-1)^2} + \frac{1}{n^2} < \dots < a_1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} = a_1 + \sum_{k=2}^{n} \frac{1}{k^2} < a_1 + \sum_{k=2}^{\infty} \frac{1}{k^2} < \infty$$

Thus $0 < a_n < a_1 + \sum_{k=2}^{\infty} \frac{1}{k^2}$, $\forall n \in \mathbb{N}^*$, a_n is bounded

Bolzano-Weierstrass: $\exists \{k_n\}_{n\in\mathbb{N}^*} \subseteq \mathbb{N}^*$, $a \in \mathbb{R}^+$: $\lim_{n\to\infty} a_{k_n} = a$, $k_n \nearrow$

We will show that the terms of a_n between consecutive of a_{k_n} also approach a. Take $\varepsilon > 0$

$\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges, hence $\exists n_0 \in \mathbb{N}^*$: $\sum_{m=n_0}^{\infty} \frac{1}{m^2} < \frac{\varepsilon}{2}$

$a_{k_n} \xrightarrow{n\to\infty} a$ hence $\exists N_0 \in \mathbb{N}^*$: $|a_{k_n} - a| < \frac{\varepsilon}{2} \Leftrightarrow a - \frac{\varepsilon}{2} < a_{k_\ell} < a + \frac{\varepsilon}{2}$ $\forall n > N_0$

Take some $n > k_{N_0}, k_{n_0}$ (also $k_{n_0} \geq n_0$), then $\exists \ell > N_0$: $k_\ell < n < k_{\ell+1}$

- $a_n < a_{n-1} + \frac{1}{n^2} < a_{n-2} + \frac{1}{(n-1)^2} + \frac{1}{n^2} < \dots < a_{k_\ell} + \frac{1}{(k_\ell+1)^2} + \dots + \frac{1}{n^2} = a_{k_\ell} + \sum_{m=k_\ell+1}^{n} \frac{1}{m^2} \overset{k_\ell+1 > k_\ell > k_{n_0} \geq n_0}{<} a_{k_\ell} + \sum_{m=n_0}^{\infty} \frac{1}{m^2} < a + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = a + \varepsilon$

- $a_{k_{\ell+1}} < a_{k_{\ell+1}-1} + \frac{1}{k_{\ell+1}^2} < \dots < a_n + \sum_{m=n+1}^{k_{\ell+1}} \frac{1}{m^2} < a_n + \sum_{m=n_0}^{\infty} \frac{1}{m^2} \Rightarrow a_n > a_{k_{\ell+1}} - \sum_{m=n_0}^{\infty} \frac{1}{m^2} > a - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = a - \varepsilon$

hence $\forall n > k_{N_0}, k_{n_0}$ $a - \varepsilon < a_n < a + \varepsilon \Leftrightarrow |a_n - a| < \varepsilon$ for every $\varepsilon > 0$

Therefore $\lim_{n\to\infty} a_n = a$

Problem 2(Day 2) (Konstantinos)

AM-GM: $\dfrac{\overbrace{a_n + a_n + \dots + a_n}^{n \text{ terms}} + \frac{1}{n+1}}{n+1} \geq \sqrt[n+1]{a_n \cdots a_n \frac{1}{n+1}} \Rightarrow$

$$\frac{na_n + \frac{1}{n+1}}{n+1} \geq \frac{a_n^{\frac{n}{n+1}}}{\sqrt[n+1]{n+1}} \Rightarrow \frac{n}{n+1} a_n + \frac{1}{(n+1)^2} \geq \frac{a_n^{\frac{n}{n+1}}}{\sqrt[n+1]{n+1}} \Rightarrow$$

$$a_n + \frac{1}{(n+1)^2} \geq \frac{n}{n+1} a_n + \frac{1}{(n+1)^2} \geq \frac{a_n^{\frac{n}{n+1}}}{\sqrt[n+1]{n+1}} > \frac{a_n^{\frac{n}{n+1}}}{2}$$

hence $\displaystyle\sum_{n=1}^{N} a_n^{\frac{n}{n+1}} < 2\left(\sum_{n=1}^{N} a_n + \sum_{n=1}^{N} \frac{1}{(n+1)^2}\right) \quad \forall N \in \mathbb{N}^*$

thus $\displaystyle\sum_{n=1}^{\infty} a_n^{\frac{n}{n+1}}$ converges

Note that $\dfrac{1}{\sqrt[n+1]{n+1}} > \frac{1}{2} \Leftrightarrow 2 > \sqrt[n+1]{n+1} \Leftrightarrow 2^{n+1} > n+1$ ✓

Problem 9(Day 2)

Assume without loss of generality that $\sum_{k=1}^{n} a_k \geq 0$

Consider $S_n = \sum_{k=1}^{n} a_k$, $S_{n-1} = \sum_{k=1}^{n-1} a_k - a_n$, ..., $S_0 = -\sum_{k=1}^{n} a_k$

$S_n \geq 0$ and if $S_n \leq \max|a_k|$ the the desired m is n,

so assume $S_n > \max|a_k|$.

Then $\{i \in \{0,1,\dots,n\} : S_i > \max|a_k|\}$ is non empty and

we can choose i_1 minimal such that $S_{i_1} > \max|a_k|$

Similarly if $S_0 \geq -\max|a_k|$ we are done, so assume

$S_0 < -\max|a_k|$, then $\{i \in \{0,1,\dots,n\} : S_i < -\max|a_k|\} \neq \emptyset$

and we can choose maximal i_2 such that $S_{i_2} < -\max|a_k|$

• Case 1: $i_1 > i_2 + 1$ $\quad \exists j \in \{0,1,\dots,n\} : i_2 < j < i_1$

$S_j < -\max|a_k|$ can't hold because i_2 is maximal, so $S_j \geq -\max|a_k|$

$S_j > \max|a_k|$ can't hold because i_1 is maximal, so $S_j \leq \max|a_k|$

hence $-\max|a_k| \leq S_j \leq \max|a_k|$, $m = j$

• Case 2: $i_1 = i_2 + 1$ $\quad S_{i_1} - S_{i_2} = \sum_{k=1}^{i_1} a_k - \sum_{k=i_1+1}^{n} a_k - \left(\sum_{k=1}^{i_2} a_k - \sum_{k=i_2+1}^{n} a_k\right) = 2a_{i_1}$

but $S_{i_1} - S_{i_2} < \max|a_k| - (-\max|a_k|) = 2\max|a_k|$

hence $2a_{i_1} < 2\max|a_k| \Rightarrow a_{i_1} < \max|a_k|$ contradiction

• Case 3: $i_2 > i_1$ Either $\exists j : i_2 > j > i_1$ with $-\max|a_k| < S_j < \max|a_k|$

or $\exists j : i_2 \geq j+1 > j \geq i_1$ with $S_{j+1} < -\max|a_k|$, $S_j > \max|a_k|$

which leads to a contradiction as in Case 2