Problem 3

We first apply the logarithm turning the product into a sum

$$A_n = \frac{\left(1^{1^p}\cdot 2^{2^p}\cdot\ldots\cdot n^{n^p}\right)^{\frac{1}{n^{p+1}}}}{n^{\frac{1}{p+1}}},$$

$$\tilde{A}_n = \ln A_n = \frac{1}{n^{p+1}}\sum_{k=1}^{n} k^p\ln k - \frac{1}{p+1}\ln n = \frac{\sum_{k=1}^{n} k^p\ln k - \frac{n^{p+1}}{p+1}\ln n}{n^{p+1}}$$

Our idea is to apply the Cesàro-Stolz lemma to simplify the sum in the numerator. We consider the following quotient

$$\frac{\sum_{k=1}^{n+1} k^p\ln k - \frac{(n+1)^{p+1}}{p+1}\ln(n+1) - \left(\sum_{k=1}^{n} k^p\ln k - \frac{n^{p+1}}{p+1}\ln n\right)}{(n+1)^{p+1}-n^{p+1}} = \frac{(n+1)^p\ln(n+1) - \frac{(n+1)^{p+1}}{p+1}\ln(n+1) + \frac{n^{p+1}}{p+1}\ln n}{(n+1)^{p+1}-n^{p+1}}$$

We need to approximate the quantities in the numerator and the denominator. We know that for large n, the logarithm grows very slowly, so $\log(n+1)\cong\log n$. Also thinking of the binomial expansion $(n+1)^{p+1}\cong n^{p+1}+(p+1)n^p$

Let's quantify these approximations using inequalities

• **Lemma 1:** $\frac{1}{n+1}\le\ln(n+1)-\ln n\le\frac{1}{n}\quad\forall n\in\mathbb{N}^*$

Define $a_n=(1+\frac{1}{n})^n$, $n\in\mathbb{N}^*$, recall that e is defined as $e=\lim_{n\to\infty}(1+\frac{1}{n})^n$

Bernoulli's inequality: $(1+\frac{1}{n+1})^{\frac{n+1}{n}}\ge 1+\frac{n+1}{n}\frac{1}{n+1}=1+\frac{1}{n}\Rightarrow a_{n+1}\ge a_n$, hence $a_n\nearrow$

thus $a_n\le\lim_{k\to\infty}a_k\Rightarrow(1+\frac{1}{n})^n\le e\Rightarrow n\ln(\frac{n+1}{n})\le\ln e\Rightarrow\ln(n+1)-\ln n\le\frac{1}{n}$

Define $b_n=(1+\frac{1}{n})^{n+1}$, $n\in\mathbb{N}^*$, again $\lim_{n\to\infty}b_n=\lim_{n\to\infty}(1+\frac{1}{n})^n\cdot\lim_{n\to\infty}(1+\frac{1}{n})=e\cdot 1=e$

$$\frac{b_{n+1}}{b_n}=\frac{(1+\frac{1}{n+1})^{n+2}}{(1+\frac{1}{n})^{n+1}}=(1+\frac{1}{n+1})\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}=(1+\frac{1}{n+1})\left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1}=(1+\frac{1}{n+1})(1-\frac{1}{(n+1)^2})^{n+1}=$$

$$\left[1+(n+1)\frac{1}{(n+1)^2}\right]\left[1-\frac{1}{(n+1)^2}\right]^{n+1}\overset{\text{Bernoulli}}{\le}\left(1+\frac{1}{(n+1)^2}\right)^{n+1}\left(1-\frac{1}{(n+1)^2}\right)^{n+1}=\left(1-\frac{1}{(n+1)^4}\right)^{n+1}<1\Rightarrow$$

$b_{n+1}<b_n$, hence $b_n\searrow$

thus $b_n\ge\lim_{k\to\infty}b_k\Rightarrow(1+\frac{1}{n})^{n+1}\ge e\Rightarrow(n+1)\ln\frac{n+1}{n}\ge\ln e\Rightarrow\ln(n+1)-\ln n\ge\frac{1}{n+1}$

• **Lemma 2:** $(p+1)n^p\le(n+1)^{p+1}-n^{p+1}\le(p+1)(n+1)^{p+1}\quad\forall n\in\mathbb{N}^*$

$p+1\ge1$, Bernoulli's inequality: $(1+\frac{1}{n})^{p+1}\ge1+\frac{p+1}{n}\Rightarrow\frac{(n+1)^{p+1}}{n^{p+1}}\ge1+\frac{p+1}{n}\Rightarrow$

$(n+1)^{p+1}\ge n^{p+1}+(p+1)n^p\Rightarrow(n+1)^{p+1}-n^{p+1}\ge(p+1)n^p$

$p+1\ge1$, $-\frac{1}{n+1}\ge-1$, Bernoulli's inequality: $(1-\frac{1}{n+1})^{p+1}\ge1-\frac{p+1}{n+1}\Rightarrow$

$\frac{n^{p+1}}{(n+1)^{p+1}}\ge1-\frac{p+1}{n+1}\Rightarrow n^{p+1}\ge(n+1)^{p+1}-(p+1)(n+1)^p\Rightarrow(p+1)(n+1)^p\ge(n+1)^{p+1}-n^{p+1}$

Similarly if $p\ge1$ $\quad pn^{p-1}\le(n+1)^p-n^p\le p(n+1)^{p-1}$

whereas if $0<p<1$ $\quad p(n+1)^{p-1}\le(n+1)^p-n^p\le pn^{p-1}$ because Bernoulli's inequality is reversed when the exponent ≤1.

We will now bound the numerator

• $(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}\ln n\ge$

$(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}(\ln(n+1)-\frac{1}{n})=$

$\left((n+1)^p-\frac{(n+1)^{p+1}-n^{p+1}}{p+1}\right)\ln(n+1)-\frac{n^p}{p+1}\ge\left((n+1)^p-\frac{(p+1)(n+1)^p}{p+1}\right)\ln(n+1)-\frac{n^p}{p+1}\ge-\frac{n^p}{p+1}$

• $(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}\ln n\le$

$(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}(\ln(n+1)-\frac{1}{n+1})=$

$\left((n+1)^p-\frac{(n+1)^{p+1}-n^{p+1}}{p+1}\right)\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}\le\left((n+1)^p-\frac{(p+1)n^p}{p+1}\right)\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}=$

$\left((n+1)^p-n^p\right)\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}\overset{p\ge1}{\le}p(n+1)^{p-1}\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}$

So far $-\frac{n^p}{p+1}\le(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}\ln n\le p(n+1)^{p-1}\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}$ (1)

We also bound the denominator

$\frac{1}{(p+1)(n+1)^p}\le\frac{1}{(n+1)^{p+1}-n^{p+1}}\le\frac{1}{(p+1)n^p}$ (2)

Before multiplying note that $p(n+1)^{p-1}\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}\xrightarrow{n\to\infty}-\infty$

hence for all large n, $p(n+1)^{p-1}\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}<0$

All terms in (1) are negative and all terms in (2) are positive

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\}(\times)\quad\frac{-\frac{n^p}{p+1}}{(p+1)n^p}\le\frac{(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}\ln n}{(n+1)^{p+1}-n^{p+1}}\le\frac{p(n+1)^{p-1}\ln(n+1)-\frac{1}{p+1}\frac{n^{p+1}}{n+1}}{(p+1)(n+1)^p}$$

$$-\frac{1}{(p+1)^2}\le\frac{(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}\ln n}{(n+1)^{p+1}-n^{p+1}}\le\frac{p}{p+1}\frac{\ln(n+1)}{n+1}-\frac{1}{(p+1)^2}\left(\frac{n}{n+1}\right)^{p+1}\xrightarrow{n\to\infty}-\frac{1}{(p+1)^2}$$

Hence $\lim_{n\to\infty}\frac{(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}\ln n}{(n+1)^{p+1}-n^{p+1}}=-\frac{1}{(p+1)^2}$ and by the Cesàro-Stolz lemma since both the numerator and the denominator tend to $\pm\infty$ and n^{p+1} is increasing $\lim_{n\to\infty}\tilde{A}_n=\lim_{n\to\infty}\frac{(n+1)^p\ln(n+1)-\frac{(n+1)^{p+1}}{p+1}\ln(n+1)+\frac{n^{p+1}}{p+1}\ln n}{(n+1)^{p+1}-n^{p+1}}=-\frac{1}{(p+1)^2}\Rightarrow\lim_{n\to\infty}\log A_n=-\frac{1}{(p+1)^2}\Rightarrow\lim_{n\to\infty}A_n=e^{-\frac{1}{(p+1)^2}}$

Problem 7

We will use the following observation: $\sqrt[3]{n}-\sqrt[3]{m}=\dfrac{n-m}{n^{\frac{2}{3}}+(nm)^{\frac{1}{3}}+m^{\frac{2}{3}}}$

We will choose n,m to be polynomials of k such that the dominant term cancels out. Choose $n_k=\alpha^3k^3+bk$, $m_k=\alpha^3k^3$, where $\alpha,b\in\mathbb{N}^*$

Then $$\sqrt[3]{n_k}-\sqrt[3]{m_k}=\frac{\alpha^3k^3+bk^2-\alpha^3k^3}{(\alpha^3k^3+bk^2)^{\frac{2}{3}}+\left[(\alpha^3k^3+bk^2)\alpha^3k^3\right]^{\frac{1}{3}}+(\alpha^3k^3)^{\frac{2}{3}}}=\frac{bk^2}{\alpha^2k^2\left(1+\frac{b}{\alpha^3}\frac{1}{k}\right)^{\frac{2}{3}}+\alpha^2k^2\left(1+\frac{b}{\alpha^3}\cdot\frac{1}{k}\right)^{\frac{1}{3}}+\alpha^2k^2}=$$

$$\frac{b}{\alpha^2\left[\left(1+\frac{b}{\alpha^3}\frac{1}{k}\right)^{\frac{2}{3}}+\left(1+\frac{b}{\alpha^3}\frac{1}{k}\right)^{\frac{1}{3}}+1\right]}\xrightarrow{k\to\infty}\frac{b}{3\alpha^2}$$

We now see that if $r\in\mathbb{Q}$, we can write $r=\frac{p}{q}$, $p,q\in\mathbb{Z}$ and choose $\alpha=q^2$, $b=3pq$

then $\sqrt[3]{(q^2)^3k^3+3pqk^2}-\sqrt[3]{(q^2)^3k^3}\xrightarrow{k\to\infty}\frac{p}{q}=r$

We will need to tweak this argument of irrationals. Take $\tilde{r}\notin\mathbb{Q}$, then

$\exists\{r_k\}_{k\in\mathbb{N}^*}\subset\mathbb{Q}:\ \lim\limits_{k\to\infty}r_k=\tilde{r}$. Write $r_k=\frac{p_k}{q_k}$, $p_k,q_k\in\mathbb{Z}$

We will use $\alpha_k=q_k^2$, $b_k=3p_kq_k$, $n_k=\alpha_k^3k^3+b_kk^2$, $m_k=\alpha_k^3k^3$

$$\sqrt[3]{n_k}-\sqrt[3]{m_k}=\frac{b_kk^2}{(\alpha_k^3k^3+b_kk^2)^{\frac{2}{3}}+\left[(\alpha_k^3k^3+b_kk^2)\alpha_k^3k^3\right]^{\frac{1}{3}}+(\alpha_k^3k^3)^{\frac{2}{3}}}=\frac{b_k}{\alpha_k^2}\frac{1}{\left(1+\frac{b_k}{\alpha_k^3}\frac{1}{k}\right)^{\frac{2}{3}}+\left(1+\frac{b_k}{\alpha_k^3}\frac{1}{k}\right)^{\frac{1}{3}}+1}\xrightarrow{k\to\infty}3\tilde{r}\,\frac{1}{3}=\tilde{r}$$

because $\lim\limits_{k\to\infty}\frac{b_k}{\alpha_k^2}=3\tilde{r}$, $\lim\limits_{k\to\infty}\alpha_k=\infty$

The second equality holds because otherwise α_k is bounded, say $\alpha_k\le M^2\Rightarrow|q_k|\le M$

and then $r_k\neq r_l$ $|r_k-r_l|=\left|\frac{p_k}{q_k}-\frac{p_l}{q_l}\right|=\frac{|p_kq_l-p_lq_k|}{|q_kq_l|}\ge\frac{1}{M!}$ which implies that $\{r_k\}$ doesnt converge.

Finally $\sqrt[3]{n_k}-\sqrt[3]{m_k}$ may tend to any $r\in\mathbb{R}$ as desired.