# Math Competition Preparation Seminar

Sequences and Series

December 1, 2025

## 1 Theory/Background

### 1.1 Arithmetic mean - Geometric mean Inequality

Let $n \in \mathbb{N}^*$ and $a_1, a_2, .., a_n \in \mathbb{R}^+$. Then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \sqrt[n]{\prod_{k=1}^{n} a_k}$$

Equality holds if and only if $a_1 = a_2 = .. = a_n$.

### 1.2 Bernoulli's Inequality

- i) $(1+x)^r \geq 1 + rx$, $\forall r \geq 1, x \geq -1$.
- ii) $(1+x)^r \leq 1 + rx$, $\forall r \in [0,1], x \geq -1$.

### 1.3 Bolzano-Weierstrass Theorem

Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\exists M > 0 : |a_n| \leq M$, $\forall n \in \mathbb{N}$. Then $\exists \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $a_{n_k}$ converges.

### 1.4 Cesáro-Stolz Lemma

Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$.

- If $a_n \to 0, b_n \to 0,$ as $n \to \infty$ and $\exists n_0 \in \mathbb{N}$ such that $b_n$ is decreasing $\forall n > n_0$, or
- If $a_n \to \infty, b_n \to \infty,$ as $n \to \infty$ and $\exists n_0 \in \mathbb{N}$ such that $b_n$ is increasing $\forall n > n_0$,

and $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in [-\infty, \infty]$, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and $\lim_{n\to\infty} \frac{a_n}{b_n} = l$.

## 2 Problems

**Problem 1** Let $\{a_n\}_{n\in\mathbb{N}^*} \subset \mathbb{R}^+$ such that $a_{n+1} < a_n + \frac{1}{(n+1)^2}$, $\forall n \in \mathbb{N}^*$. Prove that $a_n$ converges.

**Problem 2** Consider $\{a_n\}_{n\in\mathbb{N}} : a_n \in \mathbb{R}^+$, such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} {a_n}^{\frac{n}{n+1}}$ also converges.

**Problem 3** Prove the following limit for $p \geq 0$.

$$\lim_{n\to\infty} \frac{(1^{1^p} \cdot 2^{2^p} \cdots\ n^{n^p})^{1/n^{p+1}}}{n^{\frac{1}{p+1}}} = e^{-(p+1)^2}$$

**Problem 4** Calculate the following limit for $p \geq 0$.

$$\lim_{n\to\infty} [(1^{1^p} \cdot 2^{2^p} \cdots\ (n+1)^{(n+1)^p})^{1/(n+1)^{p+1}} - (1^{1^p} \cdot 2^{2^p} \cdots\ n^{n^p})^{1/n^{p+1}}]$$

**Problem 5** Let $a_n$ denote the number of '0's in the ternary representation of $n \in \mathbb{N}^*$. Find all $x > 0$ such that the following series converges

$$\sum_{n=1}^{\infty} \frac{x^{a_n}}{n^3}$$

**Problem 6** Define for $n \in \mathbb{N}$

$$x_n = min\{|a - b\sqrt{3}| : a, b \in \mathbb{N}, a + b = n\}.$$

Find the smallest $p > 0$ such that $x_n \leq p$, $\forall n \in \mathbb{N}^*$.

**Problem 7** Can $\sqrt{3}$ be written as a limit of a sequence of the form $\sqrt[3]{n} - \sqrt[3]{m}$, where $n, m \in \mathbb{N}^*$?

**Problem 8** Determine the values of $d > 0$ for which the following series $\sum_{n=1}^{\infty} sin(logn)/n^d$ converges.

**Problem 9** Prove that for a finite sequence $\{a_k\}_{k=1}^{n}$, $\exists m \in \{0, 1, \cdot\cdot, n\}$ such that

$$|\sum_{k\leq m} a_k - \sum_{m\leq k} a_k| \leq max_{k\in\{1,2,\cdot\cdot,n\}}|a_k|$$

**Problem 10** Prove that for a finite sequence $\{a_k\}_{k=1}^{n}$, with $a_k > -1$, $\forall k \in \{1, 2, \cdot\cdot, n\}$

- i) If $S = a_1 + \cdot\cdot + a_n \geq 0$, then

$$\prod_{k=1}^{n}(1 + a_k) \leq \sum_{k=0}^{n} \frac{S^k}{k!}$$

- ii) If

$$\sigma = \frac{a_1}{a_1 + 1} + \cdot\cdot + \frac{a_n}{a_n + 1} \geq 0,$$

  then

$$\prod_{k=1}^{n}(1 + a_k) \geq \sum_{k=0}^{n} \frac{\sigma^k}{k!}$$

When does equality hold in i), ii)?