Problem 1(Thanos)

Monday, November 24, 2025 11:06 PM

Since the integrant $f^2(t)+f'^2(t)$ is continuous the integral is well defined and the integral and we can differentiate the given equation to get

$$\frac{d}{dx}f^2(x)=\frac{d}{dx}\left(\int_0^x (f^2(t)+f'^2(t))\,dt+2025\right)\Rightarrow 2f(x)f'(x)=f^2(x)+f'^2(x)\Rightarrow$$

$$f^2(x)-2f(x)f'(x)+f'^2(x)=0\Rightarrow (f(x)-f'(x))^2=0\Rightarrow f(x)-f'(x)=0\Rightarrow e^{-x}f'(x)-e^{-x}f(x)=0\Rightarrow$$

$$\frac{d}{dx}\left(e^{-x}f(x)\right)=0\Rightarrow e^{-x}f(x)=C\Rightarrow f(x)=Ce^x,\quad f(0)=Ce^0=C$$

Also setting $x=0$ in the given relation

$$f^2(0)=\int_0^0 (f^2(t)+f'^2(t))\,dt+2025=0+2025=2025\Rightarrow f(0)=\pm 45$$

Therefore $f(x)=\pm 45e^x$, $x\in\mathbb{R}$

Finally substituting the functions we found back in the given relation we verify that it is indeed the solution.

$$f^2(x)=\int_0^x (f^2(t)+f'^2(t))\,dt+2025=\int_0^x \left[(45e^t)^2+\left(\frac{d}{dt}(45e^t)\right)^2\right]dt+2025=$$

$$\int_0^x 2\cdot 45^2e^{2t}\,dt+2025=45^2\int_0^x 2e^{2t}\,dt+2025=45^2e^{2t}\Big|_0^x+2025=$$

$$45^2(e^{2x}-e^0)+2025=45^2e^{2x}-45^2+2025=45^2e^{2x}\ \checkmark$$

Problem 2

Monday, November 24, 2025 11:22 PM

We will use the partial fraction decomposition of $\frac{g}{f}$

Since the x_i's are distinct, for $x \neq x_1, x_2, \dots, x_n$ we can write

$$\frac{g(x)}{f(x)} = \frac{x^{n-1}+a_{n-2}x^{n-2}+\dots+a_0}{(x-x_1)(x-x_2)\dots(x-x_n)} = \sum_{i=1}^{n} \frac{A_i}{x-x_i} \Rightarrow x\,\frac{x^{n-1}+a_{n-2}x^{n-2}+\dots+a_0}{(x-x_1)(x-x_2)\dots(x-x_n)} = x\sum_{i=1}^{n} \frac{A_i}{x-x_i} \Rightarrow$$

$$\frac{x^n+a_{n-2}x^{n-1}+\dots+a_0 x}{(x-x_1)(x-x_2)\dots(x-x_n)} = \sum_{i=1}^{n} \frac{A_i x}{x-x_i} \Rightarrow \lim_{x\to\infty} \frac{x^n+a_{n-2}x^{n-1}+\dots+a_0 x}{(x-x_1)(x-x_2)\dots(x-x_n)} = \lim_{x\to\infty} \sum_{i=1}^{n} \frac{A_i x}{x-x_i} \Rightarrow$$

$$1 = \sum_{i=1}^{n} A_i ,$$

since the polynomials on the numerator and the denominator are both of degree n and monic.

We now need to calculate the coefficients A_i. Let $j \in \{1, 2, \dots, n\}$

$$\frac{g(x)}{f(x)} = \sum_{i=1}^{n} \frac{A_i}{x-x_i} \Rightarrow (x-x_j)\frac{g(x)}{f(x)} = \sum_{i=1}^{n} (x-x_j)\frac{A_i}{x-x_i} \Rightarrow \frac{g(x)}{\frac{f(x)}{x-x_j}} = (x-x_j)\frac{A_j}{x-x_j} + \sum_{\substack{i=1\\ i\neq j}}^{n} (x-x_j)\frac{A_i}{x-x_i} \Rightarrow$$

$$\xRightarrow{f(x_j)=0} \frac{g(x)}{\frac{f(x)-f(x_j)}{x-x_j}} = A_j + \sum_{\substack{i=1\\ i\neq j}}^{n} (x-x_j)\frac{A_i}{x-x_i} \Rightarrow \lim_{x\to x_j} \frac{g(x)}{\frac{f(x)-f(x_j)}{x-x_j}} = \lim_{x\to x_j}\left(A_j + \sum_{\substack{i=1\\ i\neq j}}^{n} (x-x_j)\frac{A_i}{x-x_i}\right) \Rightarrow$$

$$\frac{g(x_j)}{f'(x_j)} = A_j + \sum_{\substack{i=1\\ i\neq j}}^{n} 0\,\frac{A_i}{x-x_i} \Rightarrow A_j = \frac{g(x_j)}{f'(x_j)}$$

Finally $\sum_{i=1}^{n} A_i = 1 \Rightarrow \sum_{i=1}^{n} \frac{g(x_j)}{f'(x_j)} = 1$

Alternatively we can use complex Analysis

Note that x_i's are simple poles of $\frac{g}{f}$

Recall that $\operatorname{Res}(f, x_j) = \lim_{z \to x_j} (z - x_j)\frac{g(z)}{f(z)} = \lim_{z \to x_j} \dfrac{g(z)}{\frac{f(z) - f(x_j)}{z - x_j}} = \dfrac{g(x_j)}{f'(x_j)}$, $j \in \{1, .., n\}$

We will integrate $\frac{g}{f}$ over the circle C_R centered at the origin with radius $R > 0$.

We will prove that this integral tends to $2\pi i$. We consider R sufficiently large.

$$\left|\int_{C_R} \frac{g(z)}{f(z)} dz - 2\pi i\right| = \left|\int_0^{2\pi} \frac{g(Re^{it})}{f(Re^{it})} R i e^{it} dt - \int_0^{2\pi} i\, dt\right| = \left| i \int_0^{2\pi} \frac{Re^{it} g(Re^{it}) - f(Re^{it})}{f(Re^{it})} dt\right| \le \int_0^{2\pi} \left|\frac{Re^{it} g(Re^{it}) - f(Re^{it})}{f(Re^{it})}\right| dt$$

Focusing on the coefficient of R^n on the numerator we see that

$$Re^{it} g(Re^{it}) - f(Re^{it}) = Re^{it}(R^{n-1} e^{i(n-1)t} + ..) - (Re^{it} - x_1)..(Re^{it} - x_n) = \cancel{R^n e^{int}} + .. - [\cancel{(Re^{it})^n} - ..]$$

hence the numerator is a polynomial of degree at most $n-1$ with respect to R, whereas the denominator is a polynomial of degree n with respect to R.

$$\left|\int_{C_R} \frac{g(z)}{f(z)} dz - 2\pi i\right| \le \int_0^{2\pi} \left|\frac{Re^{it} g(Re^{it}) - f(Re^{it})}{f(Re^{it})}\right| dt = \int_0^{2\pi} \left|\frac{\sum_{k=1}^{n-1} b_k(t) R^k}{\sum_{k=1}^{n} c_k(t) R^k}\right| dt = \int_0^{2\pi} \frac{\left|\sum_{k=1}^{n-1} b_k(t) R^k\right|}{\left|R^n + \sum_{k=1}^{n-1} c_k(t) R^k\right|} dt \le \int_0^{2\pi} \frac{\left|\sum_{k=1}^{n-1} b_k(t) R^k\right|}{R^n - \left|\sum_{k=1}^{n-1} c_k(t) R^k\right|} dt \le$$

$$\int_0^{2\pi} \frac{1}{\sqrt{R}} dt = \frac{2\pi}{\sqrt{R}} \xrightarrow{R \to \infty} 0$$, since for sufficiently large R $\sqrt{R}\left|\sum_{k=1}^{n-1} b_k(t) R^k\right| \le R^n - \left|\sum_{k=1}^{n-1} c_k(t) R^k\right|$

hence $\lim_{R \to \infty} \int_{C_R} \frac{g(z)}{f(z)} dz = 2\pi i$

Cauchy's Residue Theorem: $\int_{C_R} \frac{g(z)}{f(z)} dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, x_j) \Rightarrow \lim_{R \to \infty} \int_{C_R} \frac{g(z)}{f(z)} dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, x_j) \Rightarrow 2\pi i = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, x_j) \Rightarrow$

$\sum_{j=1}^{n} \operatorname{Res}(f, x_j) = 1 \Rightarrow \sum_{j=1}^{n} \frac{g(x_j)}{f'(x_j)} = 1$

Problem 3

Tuesday, November 25, 2025 12:51 AM

Simply applying L'hospital's rule

$$\lim_{x\to 0}\frac{\sqrt[m]{\cos x}-\sqrt[n]{\cos x}}{x^2}=\lim_{x\to 0}\frac{\frac{d}{dx}\left(\sqrt[m]{\cos x}\right)-\frac{d}{dx}\left(\sqrt[n]{\cos x}\right)}{2x}=\lim_{x\to 0}\frac{-\frac{\sin x\cdot\cos^{\frac{1}{m}-1}x}{m}+\frac{\sin x\cdot\cos^{\frac{1}{n}-1}x}{n}}{2x}=\lim_{x\to 0}\frac{\sin x}{x}\;\frac{\frac{\cos^{\frac{1}{n}-1}x}{n}-\frac{\cos^{\frac{1}{m}-1}x}{m}}{2}=1\cdot\frac{\frac{1}{n}-\frac{1}{m}}{2}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{m}\right)$$

Now let's calculate the limit without using L'hospital's rule

$$\lim_{x\to 0}\frac{\cos^{\frac{1}{m}}x-\cos^{\frac{1}{n}}x}{x^2}=\lim_{x\to 0}\frac{\left(\cos^{\frac{1}{m}}x-\cos^{\frac{1}{n}}x\right)\left[\left(\cos^{\frac{1}{m}}x\right)^{nm-1}+\left(\cos^{\frac{1}{m}}x\right)^{nm-2}\left(\cos^{\frac{1}{n}}x\right)^{2}+\dots+\left(\cos^{\frac{1}{n}}x\right)^{nm-1}\right]}{x^2\left[\left(\cos^{\frac{1}{m}}x\right)^{nm-1}+\left(\cos^{\frac{1}{m}}x\right)^{nm-2}\left(\cos^{\frac{1}{n}}x\right)^{2}+\dots+\left(\cos^{\frac{1}{n}}x\right)^{nm-1}\right]}=$$

$$\lim_{x\to 0}\frac{\left(\cos^{\frac{1}{m}}x\right)^{nm}-\left(\cos^{\frac{1}{n}}x\right)^{nm}}{x^2}\;\lim_{x\to 0}\frac{1}{\left[\left(\cos^{\frac{1}{m}}x\right)^{nm-1}+\left(\cos^{\frac{1}{m}}x\right)^{nm-2}\left(\cos^{\frac{1}{n}}x\right)^{2}+\dots+\left(\cos^{\frac{1}{n}}x\right)^{nm-1}\right]}=$$

$$\lim_{x\to 0}\frac{\cos^{n}x-\cos^{m}x}{x^2}\;\lim_{x\to 0}\frac{1}{\left[\left(\cos^{\frac{1}{m}}x\right)^{nm-1}+\left(\cos^{\frac{1}{m}}x\right)^{nm-2}\left(\cos^{\frac{1}{n}}x\right)^{2}+\dots+\left(\cos^{\frac{1}{n}}x\right)^{nm-1}\right]}=$$

$$\lim_{x\to 0}\cos^{n}x\;\lim_{x\to 0}\frac{1-\cos^{m-n}x}{x^2}\;\lim_{x\to 0}\frac{1}{\left[\left(\cos^{\frac{1}{m}}x\right)^{nm-1}+\left(\cos^{\frac{1}{m}}x\right)^{nm-2}\left(\cos^{\frac{1}{n}}x\right)^{2}+\dots+\left(\cos^{\frac{1}{n}}x\right)^{nm-1}\right]}=$$

$$\lim_{x\to 0}\cos^{n}x\cdot\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0}\left(1+\cos x+\dots+\cos^{m-n+1}\right)\;\lim_{x\to 0}\frac{1}{\left[\left(\cos^{\frac{1}{m}}x\right)^{nm-1}+\left(\cos^{\frac{1}{m}}x\right)^{nm-2}\left(\cos^{\frac{1}{n}}x\right)^{2}+\dots+\left(\cos^{\frac{1}{n}}x\right)^{nm-1}\right]}=$$

$$1^n\cdot\frac{1}{2}\overbrace{(1+1+\dots+1)}^{m-n}\frac{1}{\underbrace{1+1\cdot1+\dots+1}_{nm\text{ times}}}=\frac{1}{2}(m-n)\frac{1}{nm}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{m}\right),\quad n,m\in\mathbb{N}^*$$

We implicitly assumed $m>n$. If $n>m$ we factor out $\cos^m x$ and get the same result.

Problem 5

Tuesday, November 25, 2025 1:09 AM

First fix $y \in \mathbb{R}$ and let $x \to \pm\infty$, then $|f(x)-f(y)| \geq |x-y| \xrightarrow{x\to\pm\infty} \infty$

hence since $|f(x)| = |f(x)-f(y)+f(y)| \geq |f(x)-f(y)|+|f(y)| \xrightarrow{x\to\pm\infty} \infty$

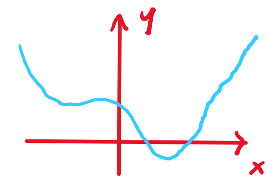
we get $\lim\limits_{x\to\pm\infty} |f(x)| = \infty$

There are 4 possibilities

$$\left\{\begin{matrix}\lim\limits_{x\to\infty} f(x)=\infty \\ \lim\limits_{x\to-\infty} f(x)=\infty\end{matrix}\right\}, \left\{\begin{matrix}\lim\limits_{x\to\infty} f(x)=\infty \\ \lim\limits_{x\to-\infty} f(x)=-\infty\end{matrix}\right\}, \left\{\begin{matrix}\lim\limits_{x\to\infty} f(x)=-\infty \\ \lim\limits_{x\to-\infty} f(x)=\infty\end{matrix}\right\}, \left\{\begin{matrix}\lim\limits_{x\to\infty} f(x)=-\infty \\ \lim\limits_{x\to-\infty} f(x)=-\infty\end{matrix}\right\}$$

☆ Assume $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \infty$

Then the graph of f will look like the figure on the right



Take some $x_0 \in \mathbb{R}$, by definition $\lim\limits_{x\to\infty} f(x)=\infty \Rightarrow \forall M\in\mathbb{R}: \exists \tilde{x}_1 \in \mathbb{R}: f(x)>M \;\; \forall x>\tilde{x}_1$

hence choosing $M=f(x_0)$, $\exists x_1 \in \mathbb{R}: f(x_1)>f(x_0),\; x_1>x_0$ (x_1 can be chosen arbitrarily large)

similarly $\lim\limits_{x\to\infty} f(x)=\infty \Rightarrow \forall M\in\mathbb{R}: \exists \tilde{x}_1 \in \mathbb{R}: f(x)>M \;\; \forall x<\tilde{x}_2$

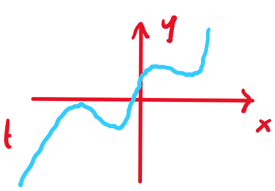
choosing $M=f(x_1)$, $\exists x_2 \in \mathbb{R}: f(x_2)>f(x_1)$, $x_2<x_0<x_1$

- f is continuous and $f(x_0)<f(x_1)<f(x_2)$ hence the Intermediate value theorem implies that $\exists x_3 \in (x_2,x_0): f(x_3)=f(x_1)$ and also $x_3\in(x_2,x_0) \Rightarrow x_2<x_3<x_0<x_1 \Rightarrow x_3<x_1$
- Now using the given inequality $|f(x_1)-f(x_3)| \geq |x_1-x_3| \Rightarrow 0 \geq |x_1-x_3|>0$ contradiction

A similar argument shows that $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = -\infty$ is impossible

- Without loss of generality suppose $\lim\limits_{x\to\infty} f(x)=\infty$, $\lim\limits_{x\to-\infty} f(x)=-\infty$

Then the graph of f will look like the figure on the right



- Take $y_0 \in \mathbb{R}$. $\lim_{x \to \infty} f(x) = \infty \Rightarrow \exists \hat{x}_1 \in \mathbb{R}: f(x) > y_0 \ \forall x > \hat{x}_1 \Rightarrow \exists x_1' \in \mathbb{R}: f(x_1') > y_0$

  Also $\lim_{x \to -\infty} f(x) = -\infty \Rightarrow \exists \hat{x}_2 \in \mathbb{R}: f(x) < y_0 \ \forall x > \hat{x}_2 \Rightarrow \exists x_2' \in \mathbb{R}: f(x_2') < y_0$, $x_2' < x_1'$

  f: continuous and $f(x_2') < y_0 < f(x_1')$ hence by the Intermediate value theorem

  $\exists \xi_0 \in (x_2', x_1'): f(\xi) = y_0 \Rightarrow y_0 \in Im(f)$

  Finally, $y_0 \in Im(f) \ \forall y_0 \in \mathbb{R} \Rightarrow Im(f) = \mathbb{R}$

Problem 6

Thursday, November 27, 2025 12:17 AM

Since f'' is bounded, $\exists K>0: |f''(x)|\le K \ \forall x\in\mathbb{R}$

Assume on the contrary that $\lim_{x\to\infty} f'(x)\neq 0$

Then $\exists \varepsilon_0>0: \forall x_0\in\mathbb{R}: \exists x>x_0: |f'(x)|\ge\varepsilon_0$

We now can define the following sequence recursively

1) Take $x_1>1$, then

2) $\exists x_2>x_1: |f'(x_2)|\ge\varepsilon_0$

3) $\exists x_3>\max(x_2,2): |f'(x_3)|\ge\varepsilon_0$

4) $\exists x_4>\max(x_3,3): |f'(x_4)|\ge\varepsilon_0$

$\vdots$

This process yields a sequence $\{x_n\}_{n\in\mathbb{N}^*}: |f'(x_n)|\ge\varepsilon_0 \ \forall n\in\mathbb{N}$

Using max ensures that $x_n>\max(x_{n-1},n-1)>x_{n-1}$, meaning that all terms are distinct and also $x_n>n-1 \Rightarrow \lim_{n\to\infty} x_n=\infty$

We can also assume without loss of generality that $f'(x_n)\ge\varepsilon_0$ by noticing that $f'(x_n)\ge\varepsilon_0$ or $f'(x_n)\le-\varepsilon_0$ (possibly both) hold infinitely many times.

Fix $n\in\mathbb{N}$ and let $x\in\mathbb{R}$. The mean value theorem implies that $\exists \psi_n$ between x_n, x such that $f''(\psi_n)=\dfrac{f'(x)-f'(x_n)}{x-x_n}$

Now $|f'(x)-f'(x_n)|=|f''(\psi_n)||x-x_n|\le K|x-x_n| \Rightarrow -K|x-x_n|\le f'(x)-f'(x_n)\le K|x-x_n| \Rightarrow$

$-K|x-x_n|\le f'(x)-f'(x_n) \Rightarrow f'(x)\ge f'(x_n)-K|x-x_n|>\varepsilon_0-K|x-x_n|$

Now choose $x\in(x_n, x_n+\frac{\varepsilon_0}{2K})$, then $x_n<x<x_n+\frac{\varepsilon_0}{2K} \Rightarrow 0<x-x_n<\frac{\varepsilon_0}{2K} \Rightarrow$

$|x-x_n|<\frac{\varepsilon_0}{2K} \Rightarrow K|x-x_n|<\frac{\varepsilon_0}{2} \Rightarrow \varepsilon_0-K|x-x_n|>\frac{\varepsilon_0}{2} \Rightarrow f'(x)>\frac{\varepsilon_0}{2} \quad \forall x\in(x_n, x_n+\frac{\varepsilon_0}{2K})$

Applying the mean value to f in the interval $(x_n, x_n+\frac{\varepsilon_0}{2K})$

yields $\exists \xi_n\in(x_n, x_n+\frac{\varepsilon_0}{2K}): f'(\xi_n)=\dfrac{f(x_n+\frac{\varepsilon_0}{2K})-f(x_n)}{x_n+\frac{\varepsilon_0}{2K}-x_n}=\dfrac{f(x_n+\frac{\varepsilon_0}{2K})-f(x_n)}{\frac{\varepsilon_0}{2K}}$

Now $|f(x_n+\frac{\varepsilon_0}{2K})-f(x_n)|=|f'(\xi_n)|\frac{\varepsilon_0}{2K}>\frac{\varepsilon_0}{2}\frac{\varepsilon_0}{2K}=\frac{\varepsilon_0^2}{4K} \Rightarrow$

$\lim_{n\to\infty}|f(x_n+\frac{\varepsilon_0}{2K})-f(x_n)|>\frac{\varepsilon_0^2}{2K} \Rightarrow |0-0|>\frac{\varepsilon_0^2}{2K} \Rightarrow 0>\frac{\varepsilon_0^2}{2K}$

because $\lim_{x_n\to\infty} x_n=\infty$, $\lim_{x\to\infty} f(x)=0$

This is a contradiction. Therefore $\lim_{x\to\infty} f'(x)=0$

Problem 9

Thursday, November 27, 2025 12:51 AM

We are going to show that $\exists x_0 \in [a,b]: f(x_0) = g(x_0)$

Since f, g are continuous they attain their maximum in $[a,b]$ by the extreme value theorem. Say $x_1, x_2 \in [a,b]$ with $f(x_1) = \max\limits_{x \in [a,b]} f(x)$, $g(x_2) = \max\limits_{x \in [a,b]} g(x)$

If $x_1 = x_2$, then simply $x_0 = x_1 = x_2$ and we are done.

Assume without loss of generality that $x_1 < x_2$ and $f(x_2) < \max\limits_{x \in [a,b]} f(x)$, $g(x_1) < \max\limits_{x \in [a,b]} g(x)$

Define $h(x) = f(x) - g(x)$, $x \in [a,b]$. Then h is continuous on $[a,b]$ and

$$h(x_1) = f(x_1) - g(x_1) = \max_{x \in [a,b]} f(x) - g(x_1) = \max_{x \in [a,b]} g(x) - g(x_1) > 0$$

$$h(x_2) = f(x_2) - g(x_2) = f(x_2) - \max_{x \in [a,b]} g(x) = f(x_2) - \max_{x \in [a,b]} f(x) < 0$$

By the Intermediate value theorem $\exists \xi$ between x_1, x_2 such that

$$h(\xi) = 0 \Rightarrow f(\xi) = g(\xi) \Rightarrow f^2(\xi) + 3f(\xi) = g^2(\xi) + 3g(\xi)$$

Problem 10

Thursday, November 27, 2025 1:01 AM

- Define $g(x) = f(x) - x, \quad x \in \mathbb{R}$

- Assume $g(x) > 0 \quad \forall x \in \mathbb{R}$. Then

1) $g(x_0) > 0 \Rightarrow f(x_0) > x_0 \quad (1)$

2) $g(f(x_0)) > 0 \Rightarrow f(f(x_0)) > f(x_0) \overset{(1)}{\Longrightarrow} f(f(x_0)) > x_0$

$\vdots$

n) $(\underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}})(x_0) > x_0$ contradiction

because x_0 is a fixed point of $\underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}$

hence $\exists x_1 \in \mathbb{R} : g(x_1) < 0$

A similar argument shows that

$\exists x_2 \in \mathbb{R} : g(x_2) < 0$

g is continuous and $g(x_2) < 0 < g(x_1)$, so the Intermediate value theorem implies that $\exists \xi$ between x_1, x_2 such that $g(\xi) = 0$

Now applying f $n-1$ times

$g(\xi) = 0 \Rightarrow f(\xi) = \xi \Rightarrow f(f(\xi)) = f(\xi) \Rightarrow f(f(\xi)) = \xi \Rightarrow$

$\ldots \Rightarrow (\underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}})(\xi) = \xi$,

ξ is a fixed point of $\underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}$, but x_0 is the unique fixed point of this function.

hence $x_0 = \xi \overset{f(\xi) = \xi}{\Longrightarrow} f(x_0) = x_0$.