# Math Competition Preparation Seminar

Continuity and derivatives

November 24, 2025

## 1 Theory/Background

### 1.1 Extreme value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then $\exists x_0 \in [a, b] : f(x) \leq f(x_0),\ \forall x \in [a, b]$.

### 1.2 Intermediate value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $\eta \in \mathbb{R}$ be between $f'(a), f'(b)$ (either $f(a) < \eta < f(b)$ or $f(a) < \eta < f(b)$). Then $\exists \xi \in (a, b) : f(\xi) = \eta$.

### 1.3 Mean value Theorem(Lagrange)

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then $\exists \xi \in (a, b)$ :

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

### 1.4 Darboux's Theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $\eta \in \mathbb{R}$ between $f'(a), f'(b)$(either $f'(a) < \eta < f'(b)$ or $f'(a) < \eta < f'(b)$). Then $\exists \xi \in (a, b) : f'(\xi) = \eta$.

## 2 Problems

**Problem 1** Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$f^2(x) = \int_0^x [f^2(t) + f'^2(t)]dt + 2025,\ \forall x \in \mathbb{R}$$

**Problem 2** Let $n \in \mathbb{N}_{n \geq 2}$ and $x_1, x_2, .., x_n, a_0, a_1, .., a_{n-2} \in \mathbb{R}$ such that $x_i \neq x_j \forall i, j \in \{1, 2, .., n\} : i \neq j$ and define $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$

$$f(x) = (x - x_1)(x - x_2)..(x - x_n)$$

$$g(x) = x^{n-1} + a_{n-2}x^{n-2} + .. + a_0$$

Show that

$$\sum_{i=1}^{n} \frac{g(x_i)}{f'(x_i)} = 1$$

**Problem 3** Compute the following limit for $m, n \in \mathbb{N}$

$$\lim_{x \to 0} \frac{\sqrt[m]{cosx} - \sqrt[n]{cosx}}{x^2}$$

**Problem 4** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and decreasing. Prove that the following system has a unique solution

$$f(x) = y, f(y) = z, f(z) = x.$$

**Problem 5** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $|f(x) - f(y)| \geq |x - y|$. Prove that $Im(f) = \mathbb{R}$.

**Problem 6** Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that $\lim_{x\to\infty} f(x) = 0$ and $f''$ is bounded. Show that $\lim_{x\to\infty} f'(x) = 0$

**Problem 7** A runner can run 6 kilometers in exactly 30 minutes. Prove that along his course he will run a single kilometer in exactly 5 minutes.

**Problem 8** Let $a, b, c \in \mathbb{R} : a < b < c$ and $f : [a, c] \to \mathbb{R}$. Prove that there exist $\xi_1, \xi_2 \in \mathbb{R} : f(a) - f(b) = f(\xi_1)(a - b), f(b) - f(c) = f(\xi_2)(b - c), \xi_1 < \xi_2$

**Problem 9** Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions satisfying $\max_{x\in[a,b]} f(x) = \max_{x\in[a,b]} g(x)$. Prove that for some $\xi \in [a, b]$

$$f^2(\xi) + 3f(\xi) = g^2(\xi) + 3g(\xi).$$

**Problem 10** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that the composition of $f$ with itself $n$ times has a unique fixed point $x_0$. Prove that $f(x_0) = x_0$.

**Problem 11** Let $f : (a, b) \to \mathbb{R}$ have finite one sided derivatives at all points in $(a, b)$. show that $f$ is differentiable in $(a, b)$ except for a countable set.

**Problem 12** Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $\exists c > 0$ such that

$$|f'(x)| \leq cf(x), \ \forall x \in \mathbb{R}.$$

i) Show that $\exists M > 0$ such that $f(x) \leq Me^{cx}$, $\forall x \in \mathbb{R}$.
ii) If $\exists \xi \in \mathbb{R} : f(\xi) = 0$, prove that $f$ is identically 0.

**Problem 13** Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that

$$f''(x) + e^{f(x)} = e, \ \forall x \in \mathbb{R}.$$

Show that $f$ cannot have a minimum value greater than 1 or a maximum value less than 1.