Math Competition Preparation Seminar

Sums

May 9, 2025

1 Theory/Background

1.1 Cauchy-Schwarz Inequality

Let $N \in \mathbb{N}^*, \{a_n\}_{n=1}^N, \{b_n\}_{n=1}^N \subset \mathbb{R}$ then

$$\left(\sum_{n=1}^{N} a_n^2\right)\left(\sum_{n=1}^{N} b_n^2\right) \geq \left(\sum_{n=1}^{N} a_n b_n\right)^2$$

1.2 base-m representation

Let $m, k \in \mathbb{N}$ with $m \geq 2$. Every $n \in \mathbb{N} : n < m^{k+1}$ can be written in a unique way in the form

$$n = a_0 + a_1 m + a_2 m^2 + \cdots + a_k m^k,$$

where $a_0, a_1, \ldots, a_k$ are integers such that $0 \leq a_i \leq m-1$ for $i \in \{0, 1, \ldots, k\}$.

2 Problems

Problem 1 Let $a_n > 0$, $\forall n \in \mathbb{N}^*$ such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ also converges.

Problem 2 Prove that the following series diverges for $\theta_0 \in (-\pi, \pi)$

$$\sum_{n=1}^{\infty} \frac{|sin(n\theta_0)|}{n}$$

Problem 3 Show that the series $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converges. If $\varepsilon > 0$, does

$$\sum_{n=1}^{\infty} \frac{(\log n)^3}{n^{1+\varepsilon}}$$

converge? Given $d \in \mathbb{N}^*$, does $\sum_{n=1}^{\infty} \frac{(\log n)^d}{n^s}$ converge for $s > 1$?

Problem 4 Show that every rational r with $0 < r < 1$ can be written as the sum of finitely many reciprocals of distinct natural numbers. Example:

$$\frac{4699}{7320} = \frac{1}{2} + \frac{1}{8} + \frac{1}{60} + \frac{1}{3660}$$

Problem 5 Let $c_n \in \mathbb{N}^*$ with $c_1 \geq 2$ satisfying

$$c_{n+1} \geq c_n(c_n - 1) + 1, \ \forall n \in \mathbb{N}^*$$

Prove that the series bellow is a rational number if and only if equality in the above inequality holds for all but finitely many values of n.

$$\sum_{n=1}^{\infty} \frac{1}{c_n}$$

Problem 6 Evaluate the sum

$$\sum_{n=1}^{\infty} \left(e - \left(1 + \frac{1}{n}\right)^n \right)$$

Problem 7 Show that the following series converges whenever $|x| > 1$ and find its sum

$$\sum_{n=0}^{\infty} \frac{2^n}{1 + x^{2^n}}$$

Problem 8 Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, with $x_1 \in (0, 1)$, such that $x_{n+1} = x_n - nx_n^2$. Prove that $\sum_{n=1}^{\infty} x_n$ converges.

Problem 9 Determine whether or not the following series converge

$$\text{i)} \ \sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{2} + .. + \sqrt{n}}$$

$$\text{ii)} \ \sum_{n=1}^{\infty} \frac{1}{1 + \sqrt[2]{2} + .. + \sqrt[n]{n}}$$

Problem 10 Evaluate the following sum for $n \in \mathbb{N}$

$$\sum_{1}^{n} \frac{1}{1^2 - 2^2 + 3^2 - .. + (-1)^{k+1}k^2}$$