Problem 1

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Taking $N \in \mathbb{N}^*$ and applying the Cauchy-Schwarz inequality yields

$$\left(\sum_{n=1}^{N} (\sqrt{a_n})^2\right) \cdot \left(\sum_{n=1}^{N} \left(\frac{1}{n}\right)^2\right) \geq \left(\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}\right)^2 \Rightarrow$$

$$\left(\sum_{n=1}^{N} a_n\right) \cdot \left(\sum_{n=1}^{N} \frac{1}{n^2}\right) \geq \left(\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}\right)^2$$

But both $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge hence $\exists M_1, M_2 > 0$ such that $\left|\sum_{n=1}^{N} a_n\right| \leq M_1$, $\sum_{n=1}^{N} \frac{1}{n^2} \leq M_2$ $\forall N \in \mathbb{N}^*$

Therefore $M_1 M_2 \geq \left(\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}\right)^2 \xRightarrow{\frac{\sqrt{a_n}}{n} > 0} \sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leq \sqrt{M_1 M_2}$ $\forall N \in \mathbb{N}^*$

Since $\frac{\sqrt{a_n}}{n} > 0$ this implies that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

Problem 2

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The issue with this problem is that i) if $\frac{\theta_0}{\pi} \in \mathbb{Q}$, $\exists m_0, n_0 \in \mathbb{N}^*: \theta_0 = \frac{m_0}{n_0}\pi$, hence for $n = kn_0$, $\sin(n\theta_0) = 0$ $\forall k \in \mathbb{N}^*$

and ii) if $\frac{\theta_0}{\pi} \notin \mathbb{Q}$, $\{\sin(n\theta_0), n \in \mathbb{N}^*\}$ is dense in $[-1,1]$, meaning $\sin(n\theta_0)$ can get arbitrarily small.

In either case we can't argue that $|\sin(n\theta_0)| \geq c \Rightarrow \sum_{n=1}^{N} \frac{|\sin(n\theta_0)|}{n} \geq c \sum_{n=1}^{N} \frac{1}{n} \xrightarrow{N\to\infty} \infty$

Solution 1

We will instead try to bound the sum of two consecutive numerators. Let $0<c<1$.

case 1: If $|\sin n\theta_0| \geq c$, then $|\sin(n+1)\theta_0| + |\sin n\theta_0| \geq c$

case 2: If $|\sin n\theta_0| < c$, then $\sin^2 n\theta_0 < c^2 \Rightarrow \cos^2 n\theta_0 > 1-c^2 \Rightarrow |\cos n\theta_0| > \sqrt{1-c^2}$

$|\sin(n+1)\theta_0| + |\sin n\theta_0| \geq |\sin(n+1)\theta_0| + |\cos\theta_0||\sin n\theta_0| \geq |\sin(n+1)\theta_0 + \cos\theta_0 \cdot \sin n\theta_0| =$

$|\cos n\theta_0 \cdot \sin\theta_0| > |\sin\theta_0|\sqrt{1-c^2}$

Thus $|\sin(n+1)\theta_0| + |\sin n\theta_0| \geq \min\{c, |\sin\theta_0|\sqrt{1-c^2}\}$ $\forall n \in \mathbb{N}^*$ $\forall c \in (0,1)$

For example we may choose $c = |\sin\theta_0|$, then

$|\sin\theta_0|\sqrt{1-c^2} = |\sin\theta_0|\sqrt{\cos^2\theta_0} = |\sin\theta_0\cos\theta_0| \geq \frac{|\sin 2\theta_0|}{2}$, $c = |\sin\theta_0| > |\sin\theta_0||\cos\theta_0| = \frac{|\sin 2\theta_0|}{2}$

hence $|\sin(n+1)\theta_0| + |\sin n\theta_0| \geq \frac{|\sin 2\theta_0|}{2}$ $\forall n \in \mathbb{N}^*$

(In this case we would have to check the cases $\sin 2\theta_0 = 0 \overset{-\pi<\theta_0<\pi}{\Longleftrightarrow} \theta_0 \in \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\}$ separately

Now that we have found $C_0 > 0: |\sin(n+1)\theta_0| + |\sin n\theta_0| \geq C_0$ we can simply say

$\max\{|\sin n\theta_0|, |\sin(n+1)\theta_0|\} \geq \frac{C_0}{2}$ and

$$\frac{|\sin n\theta_0|}{n} + \frac{|\sin(n+1)\theta_0|}{n+1} \geq \frac{\max\{|\sin n\theta_0|, |\sin(n+1)\theta_0|\}}{\max\{n, n+1\}} \geq \frac{C_0}{2(n+1)}$$

Finally, $\sum_{n=1}^{N} \frac{|\sin n\theta_0|}{n} = \sum_{n=1}^{N}\left(\frac{|\sin(2n-1)\theta_0|}{2n-1} + \frac{|\sin 2n\theta_0|}{2n}\right) \geq \sum_{n=1}^{N} \frac{C_0}{2\cdot 2n} = \frac{C_0}{4}\sum_{n=1}^{N}\frac{1}{n} \xrightarrow{N\to\infty} \infty$

hence $\sum_{n=1}^{\infty} \frac{|\sin n\theta_0|}{n}$ diverges

Observe that arguing that $|\sin n\theta_0| > c$ infinitely many times isn't sufficient. Say $\exists (n_k) \subseteq \mathbb{N}^*: |\sin n_k\theta_0| > c$ for some c then $\sum_{n=1}^{\infty}\frac{|\sin n\theta_0|}{n} \geq c\sum_{k=1}^{\infty}\frac{1}{n_k}$ doesn't imply that $\sum_{n=1}^{\infty}\frac{|\sin n\theta_0|}{n}$ diverges. We will now present an overcomplicated solution which exploits the fact that there are "linearly many" n such that $|\sin n\theta_0| > c$ for some c.

Solution 2

- Define $S_n(c) = \{k \in \mathbb{N}^*: k \leq n, |\sin k\theta_0| \leq c\}$, $n \in \mathbb{N}^*$, $0<c<1$. Assume $\lim_{n\to\infty}\frac{|S_n|}{n} = 1$ $\forall c \in (0,1)$

Suppose $m, m+1 \in S_n$, then $|\sin m\theta_0| \leq c$, $|\sin(m+1)\theta_0| \leq c$

$c(1+|\cos\theta_0|) \geq |\sin(n+1)\theta_0| + |\cos\theta_0||\sin n\theta_0| \geq |\sin(n+1)\theta_0 - \cos\theta_0\sin n\theta_0| = |\sin\theta_0\cos n\theta_0| = |\sin\theta_0||\cos n\theta_0| \Rightarrow$

$c(1+|\cos\theta_0|) \geq |\sin\theta_0||\cos n\theta_0| \Rightarrow c^2(1+|\cos\theta_0|)^2 \geq \sin^2\theta_0\cos^2 n\theta_0 \geq \sin^2\theta_0(1-c^2) \Rightarrow c^2 \geq \frac{\sin^2\theta_0}{\sin^2\theta_0 + (1+|\cos\theta_0|)^2}$ which doesn't hold $\forall c \in (0,1)$

hence $m, m+1$ can't both be in S_n

$|S_{2N}| = \left|\bigcap_{n=1}^{N}\{k \in \{2n-1, 2n\}: |\sin k\theta_0| \leq c\}\right| = \sum_{n=1}^{N}\left|\{k \in \{2n-1, 2n\}: |\sin k\theta_0| \leq c\}\right| \leq \sum_{n=1}^{N} 1 = N \Rightarrow \frac{S_{2N}}{2N} \leq \frac{1}{2}$

this contradicts the claim that $\lim_{n\to\infty}\frac{|S_n|}{n} = 1$ $\forall c \in (0,1)$, hence $\exists c_0 \in (0,1), \varepsilon_0 > 0: \forall n_0 \in \mathbb{N}^*: \exists n > n_0: \frac{|S_n|}{n} < 1 - \varepsilon_0$

- Notice that $\frac{|S_n|}{n} < 1-\varepsilon_0 \Rightarrow \frac{|\{k \in \mathbb{N}: k \leq n, |\sin k\theta_0| \leq c_0\}|}{n} < 1-\varepsilon_0 \Rightarrow \frac{n - |\{k \in \mathbb{N}: k \leq n, |\sin k\theta_0| > c_0\}|}{n} < 1-\varepsilon_0 \Rightarrow |\{k \in \mathbb{N}: k \leq n, |\sin k\theta_0| > c_0\}| > \varepsilon_0 n$

We can now define recursively $(u_n)_{n\in\mathbb{N}^*} \subset \mathbb{N}^*: |\{k \in \mathbb{N}: k \leq u_n, |\sin k\theta_0| > c_0\}| > \varepsilon_0 u_n$, $u_{n+1} > u_n$

- Since $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive integers $u_n \geq n$ and thus u_n surpasses $\frac{M}{\varepsilon_0}$ $\forall M > 0$

We can define recursively sequences $(a_n)_{n\in\mathbb{N}^*}, (m_n)_{n\in\mathbb{N}^*} \subset \mathbb{N}^*: \frac{a_n+1}{\varepsilon_0} \geq u_{m_n} > \frac{a_n}{\varepsilon_0}$

Assuming $a_1, \dots, a_n$ and $m_1, \dots, m_n$ have been defined, find $m_{n+1} \in \mathbb{N}^*$ such that $u_{m_{n+1}} > \frac{a_n+1}{\varepsilon_0}$ and choose $a_{n+1} \in \mathbb{N}^*$ (maximal) such that $u_{m_{n+1}} > \frac{a_{n+1}}{\varepsilon_0}$, then $\frac{a_{n+1}+1}{\varepsilon_0} \geq u_{m_{n+1}} > \frac{a_{n+1}}{\varepsilon_0}$

- Now notice the following:

$|\{k \in \mathbb{N}^*: k \leq u_{m_1}, |\sin k\theta_0| > c_0\}| > \varepsilon_0 u_{m_1} > \varepsilon_0\frac{a_1}{\varepsilon_0} = a_1$ so $\exists k_1, k_2, \dots, k_{a_1} \in \mathbb{N}^*: k_j \leq u_{m_1}, |\sin k_j\theta_0| > c_0$ $\forall j \in \{1, \dots, a_1\}$

$\vdots$

$|\{k \in \mathbb{N}^*: k \leq u_{m_n}, |\sin k\theta_0| > c_0\}| > \varepsilon_0 u_{m_n} > \varepsilon_0\frac{a_n}{\varepsilon_0} = a_n$ so $\exists k_{a_{n-1}+1}, \dots, k_{a_n} \in \mathbb{N}^*: k_j \leq u_{m_n}, |\sin k_j\theta_0| > c_0$ $\forall j \in \{1, \dots, a_1\}$

Finally $\sum_{n=1}^{k_{a_N}} \frac{|\sin n\theta_0|}{n} \geq \sum_{j=1}^{a_N}\frac{|\sin k_j\theta_0|}{k_j} > c_0\sum_{j=1}^{a_N}\frac{1}{k_j} = c_0\sum_{n=1}^{N}\left(\sum_{j=a_{n-1}+1}^{a_n}\frac{1}{k_j}\right) \geq c_0\sum_{n=1}^{N}\left(\sum_{j=a_{n-1}+1}^{a_n}\frac{1}{u_{m_n}}\right) \geq c_0\sum_{n=1}^{N}\frac{a_n - a_{n-1}}{u_{m_n}} \geq c_0\varepsilon_0\sum_{n=1}^{N}\frac{a_n - a_{n-1}}{a_n+1} > c_0\varepsilon_0\sum_{n=1}^{N}\frac{a_n - a_{n-1}}{2a_n} = \frac{c_0\varepsilon_0}{2}\sum_{n=1}^{N}\frac{a_n - a_{n-1}}{a_n}$

Assume $\sum_{n=1}^{N}\frac{a_n - a_{n-1}}{a_n}$ converges, then $\frac{a_n - a_{n-1}}{a_n} \xrightarrow{n\to\infty} 0 \Rightarrow$ For $\hat{\varepsilon} > 0: \exists n_0 \in \mathbb{N}^*: \left|\frac{a_n - a_{n-1}}{a_n}\right| < \hat{\varepsilon} \Rightarrow \frac{a_n - a_{n-1}}{a_n} < \hat{\varepsilon} \Rightarrow (1-\hat{\varepsilon})a_n < a_{n-1} \overset{\hat{\varepsilon}<1}{\Rightarrow} \frac{1-\hat{\varepsilon}}{a_{n-1}} < \frac{1}{a_n}$ $\forall n \geq n_0$

So now for $N > n_0$ $\sum_{n=1}^{k_{a_N}}\frac{|\sin n\theta_0|}{n} \geq \frac{c_0\varepsilon_0}{2}\sum_{n=n_0}^{N}\frac{a_n - a_{n-1}}{a_n} > \frac{c_0\varepsilon_0}{2}\sum_{n=n_0}^{N}\frac{1-\hat{\varepsilon}}{a_{n-1}}(a_n - a_{n-1}) = \frac{c_0\varepsilon_0(1-\hat{\varepsilon})}{2}\sum_{n=n_0}^{N}\frac{a_n - a_{n-1}}{a_{n-1}} = \frac{c_0\varepsilon_0(1-\hat{\varepsilon})}{2}\sum_{n=n_0}^{N}\frac{1}{a_{n-1}}\int_{a_{n-1}}^{a_n}dx = \frac{c_0\varepsilon_0(1-\hat{\varepsilon})}{2}\sum_{n=n_0}^{N}\int_{a_{n-1}}^{a_n}\frac{1}{a_{n-1}}dx \geq \frac{c_0\varepsilon_0(1-\hat{\varepsilon})}{2}\sum_{n=n_0}^{N}\int_{a_{n-1}}^{a_n}\frac{1}{x}dx =$

$\frac{c_0\varepsilon_0(1-\hat{\varepsilon})}{2}\int_{a_{n_0-1}}^{a_N}\frac{1}{x}dx \geq \frac{c_0\varepsilon_0(1-\hat{\varepsilon})}{2}(\ln a_N - \ln a_{n_0-1}) \xrightarrow[a_N\to\infty]{N\to\infty} \infty$ contradiction hence $\sum_{n=1}^{\infty}\frac{|\sin n\theta_0|}{n}$ diverges.

Problem 4

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Lemma: Let $x \in (0,1)$. Then $(a_k)_{k\in\mathbb{N}^*} \subset \mathbb{N}$: $a_k \in \{0,1\}$, $x = \sum_{k=1}^{N} \frac{a_k}{2^k} + R_N$, $0 < R_N < \frac{1}{2^N}$

We will define a_k recursively. Let $a_1 = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & x \geq \frac{1}{2} \end{cases}$, then $0 < x - \frac{a_1}{2} < \frac{1}{2}$.

Assume $a_1, \dots, a_N$ have been defined, then by induction $0 < x - \sum_{k=1}^{N} \frac{a_k}{2^k} < \frac{1}{2^N}$

Let $a_{N+1} = \begin{cases} 0, & 0 < x - \sum_{k=1}^{N} \frac{a_k}{2^k} < \frac{1}{2^{N+1}} \\ 1, & x - \sum_{k=1}^{N} \frac{a_k}{2^k} \geq \frac{1}{2^{N+1}} \end{cases}$, then $0 < x - \sum_{k=1}^{N+1} \frac{a_k}{2^k} < \frac{1}{2^{N+1}}$

Hence the lemma has been proven

Take $r \in \mathbb{Q} \cap (0,1)$, write $r = \frac{m}{n}$, $m,n \in \mathbb{N}$. Using the Lemma above

$r = \sum_{k=1}^{N} \frac{a_k}{2^k} + R_N \Leftrightarrow R_N = r - \sum_{k=1}^{N} \frac{a_k}{2^k}$, thus $R_N \in \mathbb{Q}$, so $R_N = \frac{m_N}{n_N}$

Since $R_N = r - \sum_{k=1}^{N} \frac{a_k}{2^k} = \frac{2^k \cdot m - n \sum_{k=1}^{N} a_k \cdot 2^{N-k}}{2^N n}$

we can choose $n_N = 2^N n$ (then $\gcd(m_N, n_N) = 1$ may not hold)

We now choose N large enough such that $n < 2^N$, then

$R_N < \frac{1}{2^N} \Rightarrow \frac{m_N}{n_N} < \frac{1}{2^N} \Rightarrow \frac{m_N}{2^N n} < \frac{1}{2^N} < \frac{1}{n} \Rightarrow m_N < 2^N$

Now write m_N in its binary representation

$m_N = \sum_{k=0}^{N-1} b_k \cdot 2^k$, $b_k \in \{0,1\}$ $\forall k \in \{0, \dots, N-1\}$

So $r = \sum_{k=1}^{N} \frac{a_k}{2^k} + \frac{m_N}{n_N} = \sum_{k=1}^{N} \frac{a_k}{2^k} + \frac{\sum_{k=0}^{N-1} b_k \cdot 2^k}{2^N n} = \sum_{k=1}^{N} \frac{a_k}{2^k} + \sum_{k=0}^{N-1} \frac{b_k}{2^{N-k} n}$

Finally notice the following:

- All numerators a_k, b_k are 0 or 1, so we can simply ignore those all of them that are 0.
- All denominators in $\sum_{k=1}^{N} \frac{a_k}{2^k}$ are distinct.
- All denominators in $\sum_{k=1}^{N} \frac{b_k}{2^{N-k} n}$ are distinct.

Without loss of generality denominators of the form 2^i, $2^j n$ are distinct because n can be chosen not to be a power of 2 by writing $r = \frac{3m}{3n}$.

Problem 9

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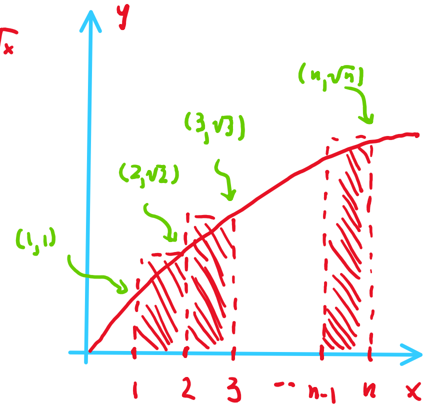
i) We can interpret the sum $1+\sqrt{2}+..+\sqrt{n}$ as the total area of rectangles of width 1, circumscribed to the curve $f(x)=\sqrt{x}$



The figure on the right suggests that

$$\sqrt{2}+..+\sqrt{n} > \int_1^n \sqrt{x}\,dx$$

This can be proven rigorously as follows

$$\sum_{k=2}^{n}\sqrt{k} = \sum_{k=2}^{n}\sqrt{k}\int_{k-1}^{k} 1dx = \sum_{k=2}^{n}\int_{k-1}^{k}\sqrt{k}\,dx \geq \sum_{k=2}^{n}\int_{k-1}^{k}\sqrt{x}\,dx = \int_1^n \sqrt{x}\,dx = \frac{2}{3}\left(n^{\frac{3}{2}}-1\right) = \frac{2n^{\frac{3}{2}}}{3}-\frac{2}{3} \Rightarrow$$

$$\sum_{k=1}^{n}\sqrt{k} \geq \frac{2n^{\frac{3}{2}}}{3}+\frac{1}{3} > \frac{2n^{\frac{3}{2}}}{3}$$, since $\sqrt{x}$ is increasing

Now $\displaystyle\sum_{n=1}^{N}\frac{1}{1+\sqrt{2}+..+\sqrt{n}} < \sum_{n=1}^{N}\frac{1}{\frac{2n^{\frac{3}{2}}}{3}} = \frac{3}{2}\sum_{n=1}^{N}\frac{1}{n^{\frac{3}{2}}}$, hence $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{2}+..+\sqrt{n}}$ converges

ii) An easy induction shows that $n<2^n \Leftrightarrow \sqrt[n]{n}<2 \quad \forall n\in\mathbb{N}^*$

Thus $\displaystyle\sum_{n=1}^{N}\frac{1}{1+\sqrt{2}+..+\sqrt[n]{n}} > \sum_{n=1}^{N}\frac{1}{2+2+..+2} = \sum_{n=1}^{N}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} \xrightarrow{N\to\infty} \infty$

hence $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{2}+..+\sqrt[n]{n}}$ diverges