Math Competition Preparation Seminar

Linear Algebra

March 28, 2025

1 Theory/Background

1.1 Characteristic Polynomial

The characteristic polynomial of a square matrix $A \in M_n(\mathbb{C})$ is defined as

$$\chi_A(\lambda) = det(\lambda I_n - A),\ \lambda \in \mathbb{C}$$

1.2 Cayley-Hamilton Theorem

Let $A \in M_n(\mathbb{C})$. Then A satisfies its characteristic polynomial

$$\chi_A(A) = \mathbb{O}_n$$

1.3 Diagonalization Theorem

Let $A \in M_n(\mathbb{C})$. If the eigenvalues of $A, \lambda_1, \lambda_2, .., \lambda_n$ are distinct then A is diagonalizable, $\exists T, D \in M_n(C)$, with T : invertible, D : diagonal, satisfying

$$A = TDT^{-1},\ \text{where } D = diag(\lambda_1, \lambda_2, .., \lambda_n)$$

1.4 Schur Triangularization Theorem

Let $A \in M_n(\mathbb{C})$. $\exists T, U \in M_n(C)$, with T : invertible, U : upper-triangular, satisfying

$$A = TUT^{-1}$$

1.5 Spectral Mapping Theorem

Let $A \in M_n(\mathbb{C})$, $\lambda_1, \lambda_2, .., \lambda_n$ the eigenvalues of A(not necessarily distinct) and P a polynomial. Then the eigenvalues of $P(A)$ are $P(\lambda_1), P(\lambda_2), .., P(\lambda_n)$.

2 Problems

Problem 1 Let $A \in M_{m\times n}(\mathbb{C}), B \in M_{n\times m}(\mathbb{C})$. Prove that

$$x^n det(xI - AB) = x^m det(xI - BA)$$

Problem 2 Consider the $2n \times 2n$ matrix

$$C_n = \begin{pmatrix} a & b & a & \dots & b \\ b & a & b & \dots & a \\ a & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & a & b & \dots & a \end{pmatrix},$$

with $a, b \in \mathbb{C}$.
Compute (a)the eigenvalues and the characteristic polynomial of C_n and (b) the determinant of the matrix obtained from C_n by replacing each diagonal element with a given $c \in \mathbb{C}$.

Problem 3 Consider the $n \times n$ matrix $A = (a_{ij})$ where $a_{ij} = 1$ if $j - i \equiv 1 \pmod{n}$, and $a_{ij} = 0$ otherwise. For $a, b \in \mathbb{R}$, find the eigenvalues of $aA + bA^T$.

Problem 4 Find all $q \in \mathbb{Q}$ for which $\exists A \in M_3(\mathbb{Q})$, such that

$$A^2 = \begin{pmatrix} q & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & q \end{pmatrix}$$

Problem 5 Let $A, B \in M_n(\mathbb{R})$, such that $AB = BA$ and $A^2 = -I_n$. Prove that $detB \geq 0$.

Problem 6 Let $A, B \in M_3(\mathbb{C})$ be invertible and satisfy $BA = A^2B$. If a, B are similar prove that $A = B = I_3$.

Problem 7 Let $A \in M_n(\mathbb{C})$ be a matrix satisfying $(AA^*)^2 = A^*A$. Prove that $AA^* = A^*A$ and that for the eigenvalues x of A, $x \neq 0 \implies |x| = 1$.

Problem 8 Let $A, B \in M_n(\mathbb{C})$, such that $B^2 = 0, A^2B + BA^2 = 2A^3$. Prove that $A^n = 0$.

Problem 9 Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be two matrices such that

$$A^2 - B^2 + x(AB - BA) = yI_n$$

where $x, y \in \mathbb{R}^*$. Prove that:

(i) $\det(A^2 - B^2) = \det(A + B)\det(A - B) = y^n$.

(ii) $(AB - BA)^n = 0$.