Problem 7(Day 4)

Monday, July 28, 2025 11:46 PM

$A^{2023} = A^*A - AA^*$ $(*)$

Let $(\lambda_i)_{i=1}^n$ be the eigenvalues of A

Take $k \in \mathbb{N}^*$: $(*) \Rightarrow A^{2023+k} = A^k A^* A - A^{k+1} A^*$

$tr(A^{2023+k}) = tr(A^k A^* A) - tr(A^{k+1} A^*) = 0 \Rightarrow$

$\lambda_1^{2023+k} + .. + \lambda_n^{2023+k} = 0 \quad \forall k \in \mathbb{N}^*$

Assume without loss of generality $\lambda_i \neq 0 \ \forall i \in \{1,..,n\}$

Assume that $\exists l \in \mathbb{N}^*: \lambda_1^m + .. + \lambda_n^m = 0 \quad \forall m \geq l$

Consider $P(\lambda) = \lambda^{l-1}(\lambda - \lambda_1)..(\lambda - \lambda_n) = \sum_{k=0}^{n} \alpha_k \lambda^{k+l-1}$

$\sum_{j=1}^{n} P(\lambda_j) = 0 \Rightarrow \sum_{j=1}^{n} \sum_{k=0}^{n} \alpha_k \lambda_j^{k+l-1} = 0 \Rightarrow \sum_{k=0}^{n} \sum_{j=1}^{n} \alpha_k \lambda_j^{k+l-1} = 0 \Rightarrow$

$\sum_{k=0}^{n} \alpha_k \sum_{j=1}^{n} \lambda_j^{k+l-1} = 0 \Rightarrow \alpha_0 \left(\sum_{j=1}^{n} \lambda_j^{l-1} \right) = 0$

but $\alpha_0 = (-1)^n \lambda_1 .. \lambda_n \neq 0$, hence $\sum_{j=1}^{n} \lambda_j^{l-1} = 0$

Therefore by induction $\sum_{j=1}^{n} \lambda_j^{0} = 0 \Rightarrow n = 0$ contradiction

So far $\lambda_j = 0 \ \forall j \in \{1,..,n\}$

$\vec{v} \in \ker A \Rightarrow A\vec{v} = \vec{0}$

$(*) \Rightarrow A^{2023}\vec{v} = A^*A\vec{v} - AA^*\vec{v} \Rightarrow AA^*\vec{v} = \vec{0} \Rightarrow$

$\vec{v}^* AA^* \vec{v} = 0 \Rightarrow (A^*\vec{v})^* A^*\vec{v} = 0 \Rightarrow \|A^*\vec{v}\|_2^2 = 0 \Rightarrow A^*\vec{v} = \vec{0} \Rightarrow \vec{v} \in \ker A^*$

$\ker A \subseteq \ker A^*$

$(*) \Rightarrow (A^*)^{2023} = A^*A - AA^*$, hence similarly $\ker A^* \subseteq \ker A$

therefore $\ker A = \ker A^*$

Let $\{u_1, u_2, .., u_k\}$ be a basis of $\ker A$ and extend it

(Steinitz's Lemma) to $\{u_1, .., u_k, u_{k+1}, .., u_n\}$, a basis of $\mathbb{C}^n$

Using Gram-Schmidt's process, we may assume that

$\{u_1, .., u_n\}$ is an orthonormal basis of $\mathbb{C}^n$

Define $P = [\vec{u}_1 .. \vec{u}_k \ \vec{u}_{k+1} .. \vec{u}_n]$, $(PP^*)_{ij} = \vec{u}_i \cdot \vec{u}_j = \delta_{ij} \Rightarrow PP^* = I_n = P^*P$

$P^*AP = P^*A[\vec{u}_1 .. \vec{u}_k \ \vec{u}_{k+1} .. \vec{u}_n] = P^*[Au_1 .. Au_k \ Au_{k+1} .. Au_n] =$

$P^*[\vec{0} .. \vec{0} \ Au_{k+1} .. Au_n] = P^*A[O_{n\times k} \ X] = (A^*P)^*[O_{n \times k} \ X] =$

$[A^*u_1 .. A^*u_k \ A^*u_{k+1} .. A^*u_n]^*[O_{n\times k} \ X] = \begin{bmatrix} O_{k\times n} \\ X^* \end{bmatrix} A [O_{n\times k} \ X] =$

$\begin{bmatrix} O \\ X^* \end{bmatrix} [O \ Y] = \begin{bmatrix} O_{k\times k} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & B \end{bmatrix}$

Finally, $P^*AP = \begin{bmatrix} O & O \\ O & B \end{bmatrix} \Rightarrow P^*A^*P = \begin{bmatrix} O & O \\ O & B^* \end{bmatrix}$

this also implies that $\text{rank}(P^*AP) = \text{rank}\left(\begin{bmatrix} O & O \\ O & B \end{bmatrix}\right) \Rightarrow \text{rank} A = \text{rank} B$

but $B \in \mathcal{M}_{(n-k)\times(n-k)}(\mathbb{C})$ and $\text{rank}(A) = n - \dim(\ker(A)) = n-k$

hence $\exists B^{-1}$

$(*)$ $P^*A^{2023}P = P^*A^*AP - P^*AA^*P \Rightarrow \begin{bmatrix} O & O \\ O & B^{2023} \end{bmatrix} = \begin{bmatrix} O & O \\ O & B^* \end{bmatrix}\begin{bmatrix} O & O \\ O & B \end{bmatrix} - \begin{bmatrix} O & O \\ O & B \end{bmatrix}\begin{bmatrix} O & O \\ O & B^* \end{bmatrix} \Rightarrow$

$B^{2023} = B^*B - BB^*$ (looking at the bottom right).

This equation is identical to the one given, so as we saw previously the eigenvalues of B are 0, which yields a contradiction, since $\exists B^{-1}$.

But what did we contradict exactly? We implicitly assumed that $k<n$ for B to be well defined, so it turns out that $k=n$

Therefore $\dim(\text{Im}(A)) = \text{rank} A = n-k = 0 \Rightarrow \text{Im}(A) = \{\vec{0}\} \Rightarrow A = O_n$

Problem 2

Tuesday, November 18, 2025 1:33 AM

Let's subtract the first row from all odd rows and the second row from all even rows

$$C_n = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & a \\ \vdots & \vdots & & \vdots \\ b & a & \cdots & a \end{bmatrix} \sim \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & a \\ & O_{2n-2 \times 2n} & & \end{bmatrix}$$

hence the geometric multiplicity of the eigenvalue 0

Thus we can be sure that $\lambda_3 = \lambda_4 = \dots = \lambda_{2n} = 0$

Also $\sum_{i=1}^{2n} \lambda_i = \operatorname{tr} C_n \Rightarrow \lambda_1 + \lambda_2 = 2na$

Consider the vector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, then $C_n \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & a \\ \vdots & \vdots & & \vdots \\ b & a & \cdots & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} na+nb \\ na+nb \\ \vdots \\ na+nb \end{bmatrix} = n(a+b) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

thus this is an eigenvector of C_n with corresponding eigenvalue $\lambda_1 = n(a+b)$

and now $\lambda_2 = 2na - \lambda_1 = n(a-b)$

The eigenvalues of C_n are $\lambda_1 = n(a+b)$, $\lambda_2 = n(a-b)$, $\lambda_3 = \lambda_4 = \dots = \lambda_{2n} = 0$

The characteristic polynomial of C_n is $\chi_{C_n}(\lambda) = \lambda^{2n-2}(\lambda - n(a+b))(\lambda - n(a-b))$

Finally $\det \begin{bmatrix} c & b & a & & b \\ a & c & b & & a \\ b & a & c & & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & a & b & \cdots & c \end{bmatrix} = \det\left(\begin{bmatrix} a & b & a & \cdots & b \\ b & a & b & \cdots & a \\ a & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & a & b & \cdots & a \end{bmatrix} - \begin{bmatrix} a-c & 0 & 0 & \cdots & 0 \\ 0 & a-c & 0 & \cdots & 0 \\ 0 & 0 & a-c & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a-c \end{bmatrix} \right) = (-1)^{2n} \det((a-c) I_{2n} - C_n)$

$$\chi_{C_n}(a-c) = (a-c)^{2n-2}(a-c-n(a+b))(a-c-n(a-b)) = (a-c)^{2n-2}(-(n-1)a - nb - c)(nb - c - (n-1)a)$$

Problem 4

Tuesday, November 18, 2025 1:49 AM

We will try to find a polynomial satisfied by A^2

$$A^4 = \begin{bmatrix} q & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & q \end{bmatrix}\begin{bmatrix} q & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & q \end{bmatrix} = \begin{bmatrix} q^2+2 & 2q+1 & 2q+1 \\ 2q+1 & q^2+2 & 2q+1 \\ 2q+1 & 2q+1 & q^2+2 \end{bmatrix} = (2q+1)\begin{bmatrix} q & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & q \end{bmatrix} + \begin{bmatrix} -q^2-q+2 & 0 & 0 \\ 0 & -q^2-q+2 & 0 \\ 0 & 0 & -q^2-q+2 \end{bmatrix} =$$

$$(2q+1)A^2 - (q^2+q-2)I$$

hence A^2 satisfies $p(x) = x^2 - (2q+1)x + (q^2+q-2)$

The minimal polynomial of A^2 divides p, hence the eigenvalues of A^2 can only be

$$\mu = \frac{2q+1 \pm \sqrt{(2q+1)^2 - 4(q^2+q-2)}}{2} = \frac{2q+1\pm\sqrt{9}}{2} = \begin{cases} q+2 & (+) \\ q-1 & (-) \end{cases}$$

but $\mathrm{tr}\, A^2 = \mu_1+\mu_2+\mu_3 \Rightarrow \mu_1+\mu_2+\mu_3 = 3q$, hence $\mu_1 = q+2$, $\mu_2=\mu_3=q-1$, $q>1$

The eigenvalues of A are $\lambda_1 = \pm\sqrt{q+2}$, $\lambda_2=\lambda_3=\pm\sqrt{q-1}$,

$$\chi_A(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - (\lambda_1+\lambda_2+\lambda_3)\lambda^2 + (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0$$

but $\chi_A(\lambda) \in \mathbb{Z}[\lambda]$ hence $\lambda_1\lambda_2\lambda_3 \in \mathbb{Q} \Rightarrow \sqrt{q+2}\,(q-1) \in \mathbb{Q} \Rightarrow \sqrt{q+2} \in \mathbb{Q}$

Cayley-Hamilton: $\chi_A(A) = O_3 \Rightarrow A^3 - \mathrm{tr}A\cdot A^2 + kA - \det A\, I_3 = O_3$

$$\left.\begin{aligned} &A^3 - \mathrm{tr}A\cdot A^2 + kA - \det A\cdot I_3 = O_3 \\ &A^4 - \mathrm{tr}A\cdot A^3 + kA^2 - \det A\cdot A = O_3 \end{aligned}\right\} \overset{(+)}{} A^4 + [k-(\mathrm{tr}A)^2]A^2 + (k\cdot\mathrm{tr}A - \det A)A - \mathrm{tr}A\cdot\det A\, I_3 = O_3 \Rightarrow$$

$$A = \frac{1}{k\cdot\mathrm{tr}A - \det A}\left(\mathrm{tr}A\cdot\det A\cdot I_3 - A^4 - [k-(\mathrm{tr}A)^2]A^2\right)$$

Problem 9

Saturday, November 22, 2025 7:31 PM

ii) • Firstly, $A^2 - B^2 + x(AB-BA) = yI_n \Rightarrow A^2 - AB + BA - B^2 + AB - BA + x(AB-BA) = yI_n \Rightarrow (A+B)(A-B) + (x+1)(AB-BA) = yI_n$ (1)

and $A^2 - B^2 + x(AB-BA) = yI_n \Rightarrow A^2 + AB - BA - B^2 - AB + BA + x(AB-BA) = yI_n \Rightarrow (A-B)(A+B) + (x-1)(AB-BA) = yI_n$ (2)

• Let $X = A-B$, $Y = A+B$, then assuming $x \neq -1$

$$\left.(x-1)(1) - (x+1)(2)\right\} \Rightarrow (x-1)YX - (x+1)XY = -2yI_n \Rightarrow XY - \frac{x-1}{x+1}YX = \frac{2y}{x+1}I_n$$

Let $\alpha = \frac{x-1}{x+1} \neq 1$, $\beta = \frac{2y}{x+1}$, then $XY - \alpha YX = \beta I_n$ (*)

• We will now try to express the characteristic polynomials of $XY-YX$ in 2 ways.

$$\chi_{XY-YX}(\lambda) = \det(\lambda I_n - (XY-YX)) \overset{(*)}{=} \det(\lambda I_n - (\alpha YX + \beta I_n - YX)) = \det((\lambda-\beta)I_n - (\alpha-1)YX) =$$
$$(\alpha-1)^n \det\left(\frac{\lambda-\beta}{\alpha-1}I_n - YX\right) = (\alpha-1)^n \chi_{YX}\left(\frac{\lambda-\beta}{\alpha-1}\right) \quad \text{(I)}$$

$$\chi_{XY-YX}(\lambda) = \det(\lambda I_n - (XY-YX)) \overset{(*)}{=} \det\left(\lambda I_n - \left(XY - \frac{XY-\beta I_n}{\alpha}\right)\right) = \det\left(\left(\lambda - \frac{\beta}{\alpha}\right)I_n - \frac{\alpha-1}{\alpha}XY\right) =$$
$$\frac{(\alpha-1)^n}{\alpha^n}\det\left(\frac{\alpha\lambda-\beta}{\alpha-1}I_n - XY\right) = \frac{(\alpha-1)^n}{\alpha^n}\chi_{XY}\left(\frac{\alpha\lambda-\beta}{\alpha-1}\right) \quad \text{(II)}$$

• Since $\chi_{XY} = \chi_{YX}$, $\left.\frac{\text{(I)}}{\text{(II)}}\right\} \Rightarrow \frac{\chi_{XY-YX}(\alpha\lambda)}{\chi_{XY-YX}(\lambda)} = \frac{(\alpha-1)^n \chi_{YX}\left(\frac{\alpha\lambda-\beta}{\alpha-1}\right)}{\frac{(\alpha-1)^n}{\alpha^n}\chi_{XY}\left(\frac{\alpha\lambda-\beta}{\alpha-1}\right)} = \alpha^n \Rightarrow$

$$\chi_{XY-YX}(\alpha\lambda) = \alpha^n \chi_{XY-YX}(\lambda) \quad \forall \lambda \in \mathbb{C}$$

Now letting $\chi_{XY-YX}(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$,

$$\chi_{XY-YX}(\alpha\lambda) = \alpha^n \chi_{XY-YX}(\lambda) \Rightarrow \sum_{k=0}^{n} a_k \alpha^k \lambda^k = \alpha^n \sum_{k=0}^{n} a_k \lambda^k \Rightarrow a_k \alpha^k = \alpha^n a_k \Rightarrow \alpha^k a_k(\alpha^{n-k}-1) = 0 \quad \forall k \in \{0,..,n\}$$

since $\alpha \neq 1$ for $k \neq n$ the above relation implies $a_k = 0$ $\forall k \in \{0,..,n-1\}$, assuming $\alpha \neq 0$

hence since $a_n = 1$, $\chi_{XY-YX}(\lambda) = \lambda^n$

• But $XY-YX=(A-B)(A+B)-(A+B)(A-B)=A^2+AB-BA-B^2-A^2+AB-BA+B^2=2(AB-BA)$

hence since the characteristic polynomial is monic $\chi_{AB-BA}(\lambda)=\chi_{XY-YX}(\lambda)=\lambda^n$

Cayley-Hamilton: $\chi_{AB-BA}(AB-BA)=O_n \Rightarrow (AB-BA)^n=O_n$

i) $\det(A^2-B^2)=\det(yI_n-x(AB-BA)) \overset{x\neq 0}{=\!=} x^n\det\left(\frac{y}{x}I_n-(AB-BA)\right)=x^n\chi_{AB-BA}\left(\frac{y}{x}\right)=x^n\left(\frac{y}{x}\right)^n=y^n$

$\det(A-B)\det(A+B)=\det((A-B)(A+B))=\det(A^2-B^2+AB-BA)=\det(yI_n-x(AB-BA)+AB-BA)=$

$\det(yI_n-(x-1)(AB-BA))=(x-1)^n\det\left(\frac{y}{x-1}I_n-(AB-BA)\right)=(x-1)^n\chi_{AB-BA}\left(\frac{y}{x-1}\right)=(x-1)^n\left(\frac{y}{x-1}\right)^n=y^n$

Finally, we have to consider the cases $x=-1$, $\alpha=0 \Leftrightarrow x=1$

• $x=1$, $A^2-B^2+AB-BA=yI_n \Rightarrow (A-B)(A+B)=yI_n \Rightarrow A-B=y(A+B)^{-1} \Rightarrow (A+B)(A-B)=yI_n \Rightarrow$

$A^2-AB+BA-B^2=yI_n$

hence $\left.\begin{array}{l} A^2-B^2+AB-BA=yI_n \\ A^2-AB+BA-B^2=yI_n \end{array}\right\} \overset{(-)}{\Rightarrow} 2(AB-BA)=O_n \Rightarrow AB=BA$

We now easily see that $(A-B)(A+B)=A^2-B^2=yI_n \Rightarrow \det(A-B)\det(A+B)=\det(A^2-B^2)=y^n$

and $(AB-BA)^n=O_n$

The proof for $x=-1$ is similar.