# Problem 1

Monday, May 26, 2025 5:58 PM

Set $n=1$, $f(1)+2f(f(1))=8$, $f(1)\in\mathbb{N}^*$ hence $f(1)$ is even

assume $f(1)\neq 2$, then $f(1)\geq 4 \Rightarrow 8\geq 4+2f(f(1)) \Rightarrow 2f(f(1))\leq 4 \Rightarrow$

$f(f(1))\leq 2 \Rightarrow f(f(1))=1$ or $f(f(1))=2$

• if $f(f(1))=1$, set $n=f(1)$

$f(f(1))+2f(f(f(1)))=3f(1)+5 \Rightarrow 1+2f(1)=3f(1)+5 \Rightarrow f(1)=-4$ $\times$

• if $f(f(1))=2$, set $n=f(1)$

$f(f(1))+2f(f(f(1)))=3f(1)+5 \Rightarrow 2+2f(2)=3\cdot f(1)+5 \Rightarrow \underbrace{2f(2)}_{\text{even}}=\underbrace{3f(1)+3}_{\text{odd}}$ $\times$

hence $f(1)=2$

Assume $\exists m\in\mathbb{N}^*(m_0=1): f(m)=m+1$

Set $n=m$ $f(m)+2f(f(m))=3m+5 \Rightarrow m+1+2f(m+1)=3m+5 \Rightarrow$

$2f(m+1)=2m+4 \Rightarrow f(m+1)=m+2$

Therefore by induction $f(n)=n+1$ $\forall n\in\mathbb{N}^*$

# Problem 2

Monday, May 26, 2025 6:17 PM

We first think of a reccurence relation for $a_n$

$a_{n-1} \equiv n-1 \pmod{k}$ and $a_{n-1}+1$ is the first

positive integer satisfying $a_{n-1}+1 \equiv n \pmod{k}$

Hence the n-th positive integer congruent to

n mod k is $a_n = a_{n-1}+1+(n-1)k \quad \forall n \in \mathbb{N}_{>1}$

Therefore, applying this relation repeatedly

$a_n = a_{n-1}+1+(n-1)k = a_{n-2}+1+(n-2)k+1+(n-1)k = ..$

$a_1+1+k+..+1+(n-2)k+1+(n-1)k = a_1+\overbrace{1+..+1}^{n-1}+k+..+(n-2)k+(n-1)k =$

$a_1+n-1+\frac{(n-1)n}{2}k \overset{a_1=1}{=\!=} n+\frac{(n-1)n}{2}k$

$a_n = n+\frac{(n-1)n}{2}k \quad \forall n \in \mathbb{N}^*$

# Problem 5

Wednesday, May 28, 2025 11:52 AM

Let $d = \gcd(x, y)$, then $x = dx'$, $y = dy'$ $x', y' \in \mathbb{N}^*$ $(x', y') = 1$

Then $d^2x'^2 - d^2y'^2 = 2x'dy'dz \Rightarrow x'^2 - y'^2 = 2x'y'z > 0$

but then $x' \mid y'^2$, so $x' = 1$, also $x' > y'$ contradiction

There are no solutions

# Problem 8

Wednesday, June 4, 2025 2:28 PM

- If $n = m^2 - 1$ $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor = \sum_{j=1}^{m-1} \sum_{j^2 \leq k < (j+1)^2} \lfloor \sqrt{k} \rfloor = \sum_{j=1}^{m-1} \sum_{j^2 \leq k < (j+1)^2} j =$

$$\sum_{j=1}^{m-1} j \left( (j+1)^2 - j^2 \right) = \sum_{j=1}^{m-1} j (2j+1) = 2 \sum_{j=1}^{m-1} j^2 + \sum_{j=1}^{m-1} j = 2 \frac{(m-1)m(2m-1)}{6} + \frac{(m-1)m}{2}$$

$$\frac{(m-1)m}{2} \left( 2 \frac{2m-1}{3} + 1 \right) = \frac{(m-1)m(4m+1)}{6}$$

- Now for $n \in \mathbb{N}$, $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor = \sum_{k=1}^{\lfloor \sqrt{n} \rfloor^2 - 1} \lfloor \sqrt{k} \rfloor + \sum_{k = \lfloor \sqrt{n} \rfloor^2}^{n} \lfloor \sqrt{k} \rfloor =$

$$\frac{(\lfloor \sqrt{n} \rfloor - 1) \lfloor \sqrt{n} \rfloor (4 \lfloor \sqrt{n} \rfloor + 1)}{6} + (n - \lfloor \sqrt{n} \rfloor^2 + 1) \lfloor \sqrt{n} \rfloor$$

# Problem 12

Friday, August 15, 2025 2:53 PM

- Let $d = \gcd(x,y)$, then we can write $x = dn$, $y = dm$, where $\gcd(n,m) = 1$

also $x^{x+y} = y^{y-x} < y^{y+x} \Rightarrow x < y$ and $n < m$

- $x^{x+y} = y^{y-x} \Leftrightarrow (dn)^{d(n+m)} = (dm)^{d(m-n)} \Leftrightarrow d^{2n} n^{n+m} = m^{m-n}$

but $n \mid d^{2n} n^{n+m} \Rightarrow n \mid m^{m-n} \xrightarrow[m>n]{\gcd(n,m)=1} n = 1$

hence $d^2 = m^{m-1}$

- Case 1: m is even: Write $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and assume $\exists i \in \{1,\dots,k\} : \alpha_i$ odd

Then in the above equation $p_i$ is raised to $\alpha_i(m-1)$, an odd exponent

which is impossible since the left hand side is a perfect square

thus $\alpha_i$ is even $\forall i$, $m$ is a perfect square.

Let $m = 2k$, $d^2 = m^{m-1} \Leftrightarrow d = (2k)^{4k^2-1}$

So far we have established that (if $m$ is even) $x^{x+y} = y^{y-x} \Rightarrow m = 2k$, $d = (2k)^{4k^2-1}$

Conversely choosing any $k \in \mathbb{N}^*$ and $m = 2k$, $d = (2k)^{4k^2-1}$, $x = d$, $y = dm$

and taking advantage of the equivalences we established (instead of one-way implications)

we see that $\left(m = 2k,\ d = (2k)^{4k^2-1},\ x = d,\ y = dm\right) \Rightarrow x^{x+y} = y^{y-x}$

(Verify this via direct substitution).

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Case 2: m is odd now $\frac{m-1}{2} \in \mathbb{N}$, so $d^2 = m^{m-1} \Rightarrow d = m^{\frac{m-1}{2}}$

Conversely, choosing any $k \in \mathbb{N}^*$, $m = 2k-1$, $d = (2k-1)^{k-1}$, $x = d$, $y = dm$

we see that $\left(m = 2k-1,\ d = (2k-1)^{k-1},\ x = d,\ y = dm\right) \Rightarrow x^{x+y} = y^{y-x}$

we see that $\left(m=2k-1,\ d=(2k-1)^{k-1},\ x=d,\ y=dm\right) \Rightarrow x^{x+y}=y^{y-x}$

To conclude $x^{x+y}=y^{y-x} \Leftrightarrow (x,y)=\left((2k)^{4k^2-1},\ (2k)^{4k^2}\right)$ or

$$(x,y)=\left((2k-1)^{2k-1},\ (2k-1)^{2k}\right)$$

for some $k \in \mathbb{N}^*$

# Problem 14

Friday, August 15, 2025 3:50 PM

- We will use $x,y$ to denote some choice of a pair among $a,b,c$, also $x \neq y$

then $x^3y - xy^3 = xy(x-y)(x+y)$ (1)

- if $x \equiv 0 \pmod 2$ or $y \equiv 0 \pmod 2$ $2 \mid xy \Rightarrow 2 \mid x^3y - xy^3$
- if $x \equiv 1 \pmod 2$ and $y \equiv 1 \pmod 2$ $2 \mid x-y \Rightarrow 2 \mid x^3y - xy^3$

hence $2 \mid x^3y - xy^3$ for any pair

- Now again using (1)

□ if $5 \mid a$ or $5 \mid b$ or $5 \mid c$ or some two integers among $a,b,c$ are equivalent mod 5

then we can find a pair $x,y$ such that $5 \mid xy(x-y) \Rightarrow 5 \mid x^3y - xy^3$

□ otherwise we consider the "nests" $\{1,4\}$, $\{2,3\}$ and notice that since $a,b,c$ are

distinct and belong to these sets (actually their remainders mod 5)

by Dirichlet's pigeonhole principle:

there exists a pair, without loss of generality $a,b$ such that

$a+b \equiv 0 \pmod 5 \Rightarrow 5 \mid a^3b - ab^3$

Finally since $2 \mid a^3b - ab^3$ (this holds for every pair), $5 \mid a^3b - ab^3$ and $2,5$ are coprime

$10 \mid a^3b - ab^3$