Math Competition Preparation Seminar
Number Theory

March 7, 2025

1 Theory/Background

1.1 Fermat's Infinite Descent Principle

There is no strictly decreasing sequence of positive integers.

1.2 Floor Function

$\lfloor \cdot \rfloor$ denotes the floor function defined as the unique integer satisfying:

$$\lfloor x \rfloor \in \mathbb{Z},\ \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1,\ \forall x \in \mathbb{R}.$$

1.3 Fundamental Theorem of Arithmetic

Every $n \in \mathbb{N} : n > 1$ can be uniquely factored as a product of primes numbers

$$n = p_1^{a_1} p_2^{a_2} .. p_k^{a_k} = \prod_{i=1}^{k} p_i^{a_i}$$

where $i \neq j \implies p_i \neq p_j,\ a_i \geq 1,\ \forall i \in \{1, 2, .., k\}$

2 Problems

Problem 1 Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(n) + 2f(f(n)) = 3n + 5,\ \forall n \in \mathbb{N}.$$

Problem 2 Let $k \in \mathbb{N}$ and consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_1 = 1$ and a_n the n-th number, greater than a_{n-1}, satisfying $a_n \equiv n (mod\, k)$. Find a closed form for a_n.

Problem 3 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(x^3 + y^3 + z^3) = f^3(x) + f^3(y) + f^3(z)$$

Problem 4 Find all positive integers a, b satisfying $ab + a + b \mid a^2 + b^2 + 1$.

Problem 5 Show that there are no $x, y, z \in \mathbb{N}^*$ satisfying

$$x^2 - y^2 = 2xyz$$

Problem 6 Show that there is no $n \in \mathbb{N} : n \mid 2^n - 1$.

Problem 7 Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Compute the sum

$$\sum_{0 \le i < j \le n} \left\lfloor \frac{x+i}{j} \right\rfloor .$$

Problem 8 Calculate $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor$ in terms of n.

Problem 9 Prove the identity

$$\sum_{k=1}^{\frac{n(n+1)}{2}} \left\lfloor \frac{-1+\sqrt{1+8k}}{2} \right\rfloor = \frac{n(n^2+2)}{3}, \quad n \ge 1.$$

Problem 10 Find the integers n for which $\frac{n^3-3n^2+4}{2n-1}$ is an integer.

Problem 11 Prove that in the product $P = 1! \cdot 2! \cdot 3! \cdots 100!$ one of the factors can be erased so that the remaining product is a perfect square.

Problem 12 Solve in positive integers the equation

$$x^{x+y} = y^{y-x}.$$

Problem 13 Show that each positive integer can be written as the difference of two positive integers having the same number of prime factors.

Problem 14 Prove that among any three distinct $a, b, c \in \mathbb{Z}$ we can find two (say a and b), such that $10 | a^3b - ab^3$.