Math Competition Preparation Seminar

Integral Inequalities and Weierstrass approximation theorem

February 21, 2025

1 Theory/Background

1.1 Cauchy-Schwarz Inequality

Let $f, g : [a, b] \to \mathbb{R}$, such that f^2, g^2 are integrable on $[a, b]$. Then fg is also integrable and

$$\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx \geq \left(\int_a^b f(x)g(x)\,dx\right)^2$$

1.2 Hölder Inequality

Let $f, g : [a, b] \to \mathbb{R}$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $|f|^p, |g|^q$ are integrable on $[a, b]$. Then $|fg|$ is also integrable and

$$\left(\int_a^b f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)\,dx\right)^{\frac{1}{q}} \geq \int_a^b |f(x)g(x)|\,dx$$

1.3 Chebyshev Inequality

Let $f, g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and monotone functions with the same monotonicity. Then fg is integrable and

$$\int_a^b f(x)\,dx \int_a^b g(x)\,dx \leq (b - a)\left(\int_a^b f(x)g(x)\,dx\right)^2$$

. The inequality is reversed if f, g have opposite monotonicity.

1.4 Weierstrass Approximation Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. $\forall \epsilon > 0, \exists p : [a, b] \to \mathbb{R}$ polynomial, satisfying

$$|f(x) - p(x)| < \epsilon, \forall x \in [a, b]$$

In other words there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ that converges uniformly to f on $[a, b]$.

2 Problems

Problem 1 Find all continuous functions $f : [0,1] \to \mathbb{R}$ satisfying

$$\int_0^1 x^n f(x)\, dx = 0, \quad \forall n \in \mathbb{N}$$

What if the above relation only holds for all even integers?

Problem 2 A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ is called "Devin" if for every continuous function $f : [0,1] \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)\, dx.$$

Prove that $\{x_n\}$ is Devin if and only if for every nonnegative integer k it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i^k = \frac{1}{k+1}. \tag{3.3.2}$$

Problem 3 Let $f : [0,1] \to \mathbb{R}$ be continuous and satisfy $\int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx = 1$ Show that $\int_0^1 f^2(x)\, dx \geq 4$.

Problem 4 Find the maximum value of the expression

$$\frac{\left(\int_0^4 f(x)\, dx\right)^4}{\int_0^4 f(x)^4\, dx}$$

Which functions maximizes this expression?

Problem 5 Let $f : [0,+\infty) \to \mathbb{R}$ be strictly increasing with a continuous derivative and $f(0) = 0$. Let f^{-1} denote the inverse of f. Prove that for all $a, b \in \mathbb{R} : b < f(a)$

$$\int_0^a f(x)\, dx + \int_0^b f^{-1}(x) > ab$$

What if instead f is only strictly increasing?

Problem 6 Define $D_N(\theta) = \sum_{n=-N}^{n=N} e^{in\theta}$, $N \in \mathbb{N}$. Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta = \frac{4}{\pi^2} log N + O(1)$$

Problem 7 Let $f : [0,1] \to \mathbb{R}$. Suppose that f is increasing on $[0,1/2]$ and that $f(1-x) = f(x)$ for every $x \in [0,1]$. If $\phi : [0,1] \to \mathbb{R}$ is a convex function, prove that

$$\int_0^1 \phi(x) f(x)\, dx \leq \int_0^1 \phi(x)\, dx \cdot \int_0^1 f(x)\, dx.$$