Problem 1

Monday, May 19, 2025 7:20 PM

Let q be a polynomial, then $q(x)=\sum_{n=0}^{N}\alpha_n x^n$

Then $\int_0^1 f(x)q(x)dx=\sum_{n=0}^{N}\alpha_n\int_0^1 x^n f(x)dx=0$

Hence $\int_0^1 f(x)q(x)dx=0$ for all polynomials $q(x)$

Also, f: continuous on $[0,1]\Rightarrow\exists M>0: |f(x)|\le M \quad \forall x\in[0,1]$

Take $\varepsilon>0$, by the Weierstrass approximation Theorem:

$\exists P$: polynomial such that $|f(x)-P(x)|<\frac{\varepsilon}{M} \quad \forall x\in[0,1]$

Therefore $\left|\int_0^1 f(x)(f(x)-P(x))dx\right|\le\int_0^1|f(x)||f(x)-P(x)|dx\le\int_0^1 M\frac{\varepsilon}{M}dx=\varepsilon$

but $\int_0^1 f(x)(f(x)-P(x))dx=\int_0^1 f^2(x)dx-\int_0^1 f(x)P(x)dx=\int_0^1 f^2(x)dx$

hence $\int_0^1 f^2(x)dx<\varepsilon \quad \forall\varepsilon>0\Rightarrow\int_0^1 f^2(x)dx=0$

since f is continuous $f\equiv0$

In the case when $\int_0^1 x^n f(x)dx=0$ only holds for all even integers n, we see that $\int_0^1 f(x)q(x)dx=0$ whenever q is a polynomial consisting of even powers of x ($x^{2k}, k\in\mathbb{N}$)

We notice that $f(\sqrt{x}):[0,1]\to\mathbb{R}$ is well defined, since $[0,1]\xrightarrow{\sqrt{\cdot}}[0,1]\xrightarrow{f}\mathbb{R}$ and is in fact continuous as the composition of continuous functions.

By the Weierstrass approximation Theorem:

$\forall\varepsilon>0 \quad \exists P$: polynomial such that $|f(\sqrt{x})-P(x)|<\frac{\varepsilon}{M} \quad \forall x\in[0,1]$

Now choose $y\in[0,1]$, then $y^2\in[0,1]$, $|f(y)-P(y^2)|<\frac{\varepsilon}{M}$

hence $|f(y)-P(y^2)|<\frac{\varepsilon}{M} \quad \forall y\in[0,1]$

As in the previous proof

$\int_0^1 f^2(y)dy=\left|\int_0^1 f(y)(f(y)-P(y^2))dy\right|<\varepsilon \quad \forall\varepsilon>0\Rightarrow f\equiv0.$

Problem 3

Tuesday, May 20, 2025 1:46 PM

Since the integrals $\int_0^1 f(x)dx$, $\int_0^1 xf(x)\,dx$ are known and we are looking for a lower bound of $\int_0^1 f^2(x)\,dx$ we will try the following:

$$\int_0^1 (f(x)-\alpha x-\beta)^2 dx \geq 0 \Rightarrow \int_0^1 f^2(x)\,dx - 2\int_0^1 (\alpha x f(x)+\beta f(x))\,dx + \int_0^1 (\alpha x+\beta)^2 dx \geq 0 \Rightarrow$$

$$\int_0^1 f^2(x)dx \geq 2\alpha+2\beta - \frac{(\alpha+\beta)^3-\beta^3}{3\alpha} = -\frac{\alpha^2}{3} - \alpha\beta - \beta^2 + 2\alpha + 2\beta$$

We will now maximize the right hand side with respect to α, β

$$\left.\begin{array}{l} \frac{\partial}{\partial\alpha}\left(-\frac{\alpha^2}{3} - \alpha\beta - \beta^2 + 2\alpha + 2\beta\right) = 0 \\ \frac{\partial}{\partial\beta}\left(-\frac{\alpha^2}{3} - \alpha\beta - \beta^2 + 2\alpha + 2\beta\right) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -\frac{2\alpha}{3} - \beta + 2 = 0 \\ -\alpha - 2\beta + 2 = 0 \end{array}\right\} \Rightarrow \begin{bmatrix} \frac{2}{3} & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\frac{2}{3}2 - 1\cdot 1} \begin{bmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\frac{1}{3}} \begin{bmatrix} 2 \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

plugging in these values

$$\int_0^1 f^2(x)dx \geq -\frac{6^2}{3} - 6\cdot(-2) - (-2)^2 + 2\cdot 6 + 2(-2) = 4$$

Problem 4

Tuesday, May 20, 2025 4:33 PM

Apply Cauchy-Schwarz twice

$$\int_0^4 (f^2(x))^2 dx \cdot \int_0^4 1^2 dx \geq \left(\int_0^4 f^2(x) dx\right)^2 \Rightarrow \int_0^4 f^4(x) dx \cdot 4 \geq \left(\int_0^4 f^2(x) dx\right)^2$$

$$\int_0^4 f^2(x) dx \int_0^4 1^2 dx \geq \left(\int_0^4 f(x) dx\right)^2 \Rightarrow \int_0^4 f^2(x) dx \cdot 4 \geq \left(\int_0^4 f(x) dx\right)^2$$

hence $\dfrac{\left(\int_0^4 f(x) dx\right)^4}{\int_0^4 f^4(x) dx} \leq \dfrac{\left(4\int_0^4 f^2(x) dx\right)^2}{\int_0^4 f^4(x) dx} \leq 4^2 \dfrac{4\int_0^4 f^4(x) dx}{\int_0^4 f^4(x) dx} = 64$

We could instead apply Hölder's inequality:

$$\left(\int_0^4 |f(x)|^4 dx\right)^{\frac{1}{4}} \left(\int_0^4 1^q dx\right)^{\frac{1}{q}} \geq \int_0^4 |f(x) \cdot 1| dx \Rightarrow$$

$$\left(\int_0^4 f^4(x) dx\right)^{\frac{1}{4}} \cdot 4^{\frac{1}{q}} \geq \int_0^4 |f(x)| dx \geq \left|\int_0^4 f(x) dx\right| \Rightarrow$$

$$\frac{\left(\int_0^4 f(x) dx\right)^4}{\int_0^4 f^4(x) dx} \leq 4^{\frac{4}{q}}, \quad \frac{1}{4} + \frac{1}{q} = 1 \Rightarrow q = \frac{4}{3}$$

So far $\dfrac{\left(\int_0^4 f(x) dx\right)^4}{\int_0^4 f^4(x) dx} \leq 64$, 64 is an upper bound

This is in fact the maximum value, since equality in C-S, Hölder inequalities holds when $f(x) = c \cdot 1 = c$ and by substituting $f(x) = c$ the expressions attains the value 64.

Comment: Notice that applying Hölder's inequality solves the problem for any exponent (not only 4), whereas repeated applications of the C-S inequality works out only when the exponent is a power of 2.

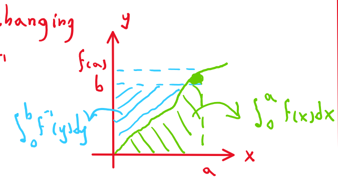
Problem 5

Tuesday, May 20, 2025 6:13 PM

First think of the graph of f.

$f(x)=y \iff f^{-1}(y)=x$ which means that by exchanging the x,y axis we get the graph of f^{-1}

The figure on the right shows that the sum of the two integrals is larger than the area of a rectangle with sides a, b.



This is exactly what we want to prove, let's make our idea precice:

$$\int_0^a f(x)\,dx+\int_0^b f^{-1}(y)\,dy \overset{y=f(x)}{\underset{dy=f'(x)dx}{=\!=}} \int_0^a f(x)\,dx+\int_{f^{-1}(0)}^{f^{-1}(b)} x f'(x)\,dx=$$

$$\int_0^a f(x)\,dx+\int_0^{f^{-1}(b)} x f'(x)\,dx=\int_0^{f^{-1}(b)} f(x)\,dx+\int_{f^{-1}(b)}^{a} f(x)\,dx+\int_0^{f^{-1}(b)} x f'(x)\,dx=$$

$$\int_0^{f^{-1}(b)}\left(f(x)+xf'(x)\right)dx+\int_{f^{-1}(b)}^{a} f(x)\,dx \quad f\nearrow \iff f^{-1}\nearrow$$

$$>$$

$$f^{-1}(b)<x<a$$

$$b<f(x)<f(a)$$

$$xf(x)\Big|_0^{f^{-1}(b)}+\int_{f^{-1}(b)}^{a} b\,dx=$$

$$xf(x)\Big|_0^{f^{-1}(b)}+bx\Big|_{f^{-1}(b)}^{a}=bf^{-1}(b)+ab-bf^{-1}(b)=ab$$

If f is merely strictly increasing (monotonicity implies integrability), according to Lebesgue's theorem, f is differentiable on $(0,a)$ except for a set of measure 0. Hence our proof still works out by excluding this set when integrating.

Problem 6

Friday, May 23, 2025 12:22 PM

First we find a closed form for Dirichlet's kernel

$$D_N(\theta)=\sum_{n=-N}^{N} e^{in\theta}=e^{-iN\theta}\sum_{n=-N}^{N} e^{i(n+N)\theta}=e^{-iN\theta}\sum_{n=0}^{2N} e^{in\theta}=e^{-iN\theta}\frac{1-e^{i(2N+1)\theta}}{1-e^{i\theta}}=$$

$$\frac{e^{-i(N+\frac{1}{2})\theta}}{e^{-i\frac{\theta}{2}}}\,\frac{1-e^{i(2N+1)\theta}}{1-e^{i\theta}}=\frac{e^{-i(N+\frac{1}{2})\theta}-e^{i(N+\frac{1}{2})\theta}}{e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}}=\frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}\qquad \forall N\in\mathbb{N}$$

We now need to find an upper and a lower bound for its norm

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|\,d\theta=\frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}\right|d\theta=\frac{1}{\pi}\sum_{k=0}^{N-1}\int_{\frac{k}{N}\pi}^{\frac{k+1}{N}\pi}\frac{|\sin(N+\frac{1}{2})\theta|}{\sin\frac{\theta}{2}}\,d\theta \qquad \underline{\underline{\varphi=\theta-\frac{k}{N}\pi}}$$

$$\frac{1}{\pi}\sum_{k=0}^{N-1}\int_0^{\frac{\pi}{N}}\frac{|\sin(N+\frac{1}{2})(\varphi+\frac{k}{N}\pi)|}{\sin\frac{\varphi+\frac{k}{N}\pi}{2}}\,d\varphi=\frac{1}{\pi}\sum_{k=0}^{N-1}\int_0^{\frac{\pi}{N}}\frac{|\sin(N+\frac{1}{2})(\varphi+\frac{k}{N}\pi)|}{\sin\frac{\varphi+\frac{k}{N}\pi}{2}}\,d\varphi\geq$$

$$\frac{1}{\pi}\sum_{k=0}^{N-1}\int_0^{\frac{\pi}{N}}\left|\frac{\sin(N+\frac{1}{2})(\varphi+\frac{k}{N}\pi)}{\frac{\varphi+\frac{k}{N}\pi}{2}}\right|d\varphi\geq\frac{1}{\pi}\sum_{k=0}^{N-1}\frac{2}{\frac{k+1}{N}\pi}\int_0^{\frac{\pi}{N}}|\sin(N+\tfrac{1}{2})(\varphi+\tfrac{k}{N}\pi)|\,d\varphi=$$

$$\frac{2N}{\pi^2}\sum_{k=0}^{N-1}\frac{1}{k+1}\int_0^{\frac{\pi}{N}}\left|\sin\left((N+\tfrac{1}{2})\varphi+\tfrac{k}{2N}\pi\right)\right|d\varphi=$$

since $(N+\frac{1}{2})\varphi+\frac{k}{2N}\pi\leq\pi \Leftrightarrow \varphi\leq\frac{2N-k}{(N+\frac{1}{2})2N}\pi=\phi_k<\frac{\pi}{N+\frac{1}{2}}<\frac{\pi}{N}$

$$\frac{2N}{\pi^2}\sum_{k=0}^{N-1}\frac{1}{k+1}\left(\int_0^{\phi_k}\sin\left((N+\tfrac{1}{2})\varphi+\tfrac{k}{2N}\pi\right)d\varphi-\int_{\phi_k}^{\frac{\pi}{N}}\sin\left((N+\tfrac{1}{2})\varphi+\tfrac{k}{2N}\pi\right)d\varphi\right)=$$

$$\frac{2N}{\pi^2}\sum_{k=0}^{N-1}\left(\frac{1}{k+1}\left[-\frac{1}{N+\frac{1}{2}}\cos\left((N+\tfrac{1}{2})\varphi+\tfrac{k}{2N}\pi\right)\right]_0^{\phi_k}-\left[-\frac{1}{N+\frac{1}{2}}\cos\left((N+\tfrac{1}{2})\varphi+\tfrac{k}{2N}\pi\right)\right]_{\phi_k}^{\frac{\pi}{N}}\right)=$$

$$\frac{2N}{\pi^2(N+\frac{1}{2})}\sum_{k=0}^{N-1}\frac{1}{k+1}\left(\cos\frac{k\pi}{2N}-\cos\pi-\cos\pi+\cos\left((N+\tfrac{1}{2})\tfrac{\pi}{N}+\tfrac{k}{2N}\pi\right)\right)=$$

$$\frac{2N}{\pi^2(N+\frac{1}{2})}\sum_{k=0}^{N-1}\frac{1}{k+1}\Big(2+\underbrace{\cos\frac{k\pi}{2N}-\cos\frac{k+1}{2N}\pi}_{\geq 0}\Big)>\frac{4N}{\pi^2(N+\frac{1}{2})}\sum_{k=0}^{N-1}\frac{1}{k+1}=\frac{4N}{\pi^2(N+\frac{1}{2})}\log N=$$

$$\frac{4N+2-2}{\pi^2(N+\frac{1}{2})}\log N=\frac{4}{\pi^2}\log N-\frac{2}{\pi^2}\frac{\log N}{N+\frac{1}{2}}$$

$\to$ bounded

We previously used $1+\frac{1}{2}+\dots+\frac{1}{N}>\log N$

Now for the upper bound notice

$$\frac{\sin(N+\frac{1}{2})\theta}{\sin\theta}=\frac{\sin N\theta\cdot\cos\frac{\theta}{2}+\cos N\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\frac{\sin N\theta}{\tan\frac{\theta}{2}}+\cos N\theta$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|\,d\theta=\frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}\right|d\theta\leq\frac{1}{\pi}\left(\int_0^{\pi}\left|\frac{\sin N\theta}{\tan\frac{\theta}{2}}\right|d\theta+\int_0^{\pi}|\cos N\theta|\,d\theta\right)$$

here $\int_0^{\pi}|\cos N\theta|\,d\theta\leq\int_0^{\pi}d\theta=\pi$

and $\int_0^{\pi}\left|\frac{\sin N\theta}{\tan\frac{\theta}{2}}\right|d\theta \overset{\tan x>x,\ x\in[0,\frac{\pi}{2})}{\leq} \int_0^{\pi}\frac{|\sin N\theta|}{\frac{\theta}{2}}\,d\theta=$

$$\sum_{k=1}^{N-1}\int_{\frac{k}{N}\pi}^{\frac{k+1}{N}\pi}\frac{|\sin N\theta|}{\frac{\theta}{2}}\,d\theta+\int_0^{\frac{\pi}{N}}\frac{|\sin N\theta|}{\frac{\theta}{2}}\,d\theta \underset{\frac{|\sin x|}{|x|}\leq 1}{\overset{\theta>\frac{k}{N}\pi}{\leq}} \sum_{k=1}^{N-1}\int_{\frac{k}{N}\pi}^{\frac{k+1}{N}\pi}\frac{|\sin N\theta|}{\frac{k\pi}{2N}}\,d\theta+\int_0^{\frac{\pi}{N}}2N\,d\theta=$$

$$\frac{2N}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\int_{\frac{k}{N}\pi}^{\frac{k+1}{N}\pi}|\sin N\theta|\,d\theta+2\pi=\frac{2N}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\int_0^{\frac{\pi}{N}}|\sin N(\varphi+\tfrac{k}{N}\pi)|\,d\varphi+2\pi=$$

$$\frac{2N}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\int_0^{\frac{\pi}{N}}|\sin(N\varphi+k\pi)|\,d\varphi+2\pi=\frac{2N}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\int_0^{\frac{\pi}{N}}|\sin(N\varphi)|\,d\varphi+2\pi=$$

$$\frac{2N}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\int_0^{\frac{\pi}{N}}\sin N\varphi\,d\varphi+2\pi=\frac{2N}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\left(-\frac{\cos N\varphi}{N}\right)\Bigg|_0^{\frac{\pi}{N}}+2\pi=\frac{2N}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\frac{2}{N}+2\pi=$$

$$\frac{4}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}+2\pi<\frac{4}{\pi}\log N+2\pi$$

Combining the above results $\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|\,d\theta\leq\frac{4}{\pi^2}\log N+3$

Finally $\frac{4}{\pi^2}\log N-B\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|\,d\theta\leq\frac{4}{\pi^2}\log N+3 \Rightarrow$

$$\int_{-\pi}^{\pi}|D_N(\theta)|\,d\theta=\frac{4}{\pi^2}\log N+O(1)$$

Note that we used the inequality $1+\frac{1}{2}+\dots+\frac{1}{N-1}<\log N$

(this inequality is inverted when we also add $\frac{1}{N}$)