Math Competition Preparation Seminar

Linear Algebra

December 19, 2024

1 Theory/Background

1.1 Main Ideas

- The determinant is multiplicative and remains constant when performing row operations.
- When proving a relation/property that holds for all matrices, prove it assuming that the matrices are invertible and then adapt your argument for non-invertible matrices.

1.2 Propositions

- The trace map is linear and satisfies $tr(AB) = tr(BA)$.
- $Ax = 0 \Rightarrow x = 0$ is equivalent to A being invertible.
- $\exists S$: invertible, such that$A = SBS^{-1}$ implies that the (square) matrices A, B have the same eigenvalues with the same respective multiplicities.
- If $\lambda_1, \lambda_2, .., \lambda_n$ are the eigenvalues of the $n \times n$ matrix A, then $\lambda_1^m, \lambda_2^m, .., \lambda_n^m$ are the eigenvalues of A^m.

2 Problems

Problem 1 Find all maps $f : \mathbb{C}^{n\times n} \to \mathbb{C}$ satisfying

- $f(cA) = cf(A)$
- $f(A + B) = f(A) + f(B)$
- $f(AB) = f(BA)$

for all $A, B \in \mathbb{C}^{n\times n}$ and $c \in \mathbb{C}$.

Problem 2 Let $A \in \mathbb{C}^{n\times n}$, satisfying $A + A^T = \mathbb{O}_n$. Prove that

$$\det(\mathbb{I}_n + \lambda A^2) \geq 0, \forall \lambda \in \mathbb{R}$$

Problem 3 Let $A, B \in \mathbb{R}^{2\times 2}$ satisfy $(AB - BA)^n = \mathbb{I}_2$ for some $n \in \mathbb{N}^*$. Prove that $(AB - BA)^4 = \mathbb{I}_2$.

Comment:Try proving the above using as elementary tools as possible.

Problem 4 Let $A, B, C, D \in \mathbb{C}^{n\times n}$, such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$$

Problem 5 Let $A, B \in \mathbb{C}^{n\times n}$. Prove that the matrices AB, BA have the same eigenvalues with the same respective multiplicities(same spectrum).

Problem 6 Prove that the following matrix is invertible.

$$A_n = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

Also, prove that the entries of it's inverse sum to n^2.

Problem 7 Find all $A \in \mathbb{C}^{n\times n}$ satisfying

$$A^{2023} = A^*A - AA^*$$

Problem 8 Let $A, B \in \mathbb{C}^{n\times n}$. If there exist $a, b \in \mathbb{C}^*$ such that $AB = aA + bB$, show that A, B commute.

Problem 9 Prove that the matrix

$$\begin{bmatrix} sin(\alpha) & sin(2\alpha) & \dots & sin(n\alpha) \\ sin(2\alpha) & sin(4\alpha) & \dots & sin(2n\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ sin(n\alpha) & sin(2n\alpha) & \dots & sin(n^2\alpha) \end{bmatrix}$$

is invertible, where $\alpha = \frac{\pi}{n+1}, n \in \mathbb{N}$.

Problem 10 If $A, B \in \mathbb{C}^{n\times n}$. Prove the following:

- If $AB = A + B$, then $AB = BA$.
- If $P \in \mathbb{C}^{n\times n}$ is invertible, such that $A = PB(A + P)$, then $ABP = PBA$.
- Does the above hold when A, B don't commute?