Problem 1

Παρασκευή, 3 Ιανουαρίου 2025 1:07 πμ

Define $E_{mk} \in \mathbb{C}^{n\times n} : (E_{mk})_{ij} = \begin{cases} 1 & i=m,\ j=k \\ 0 & \text{otherwise} \end{cases}$

Observe that $E_{mk} E_{ij} = \begin{cases} E_{mj} & , k=i \\ O_n & , k\neq i \end{cases}$

$f(cA) = c f(A) \xRightarrow[c\neq 1]{A=O_n} f(O_n) = c f(O_n) \Rightarrow f(O_n) = 0$

Choose $m, k, \ell \in \{1, 2, \dots, n\}$:

when $m \neq \ell$ $\quad f(E_{mk} E_{k\ell}) = f(E_{k\ell} E_{mk}) \Rightarrow f(E_{m\ell}) = f(O_n) = 0 \quad (1)$

when $m = \ell$ $\quad f(E_{mk} E_{km}) = f(E_{km} E_{mk}) \Rightarrow f(E_{mm}) = f(E_{kk}) \quad \forall m, k \quad (2)$

Let $f(E_{11}) = z \in \mathbb{C}$

Applying linearity

$f(A) = \sum_{i,j=1}^{n} a_{ij} f(E_{ij}) \overset{(1)}{=} \sum_{i=1}^{n} a_{ii} f(E_{ii}) \overset{(2)}{=} z \sum_{i=1}^{n} a_{ii} = z\, tr(A) \quad \forall A \in \mathbb{C}^{n\times n}$

Problem 3

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We can approach this problem using advanced Linear Algebra tools that we will examine more thoroughly in a future lesson.

Solution 1: Let $X = AB - BA$, $\operatorname{tr}X = 0$

$X^n = I_2 \Rightarrow \det(X^n) = 1 \Rightarrow (\det X)^n = 1 \xrightarrow{X \in \mathbb{R}^{2\times 2} \Rightarrow \det X \in \mathbb{R}} \det X = \pm 1$

note that if λ_1, λ_2: eigenvalues of X, then

$\lambda_1 + \lambda_2 = \operatorname{tr}X$, $\lambda_1 \lambda_2 = \det X$

Cayley-Hamilton: $\chi_X(X) = O_2 \Rightarrow X^2 - (\lambda_1 + \lambda_2)X + \det X \cdot I_2 = O_2 \Rightarrow$

$X^2 - \operatorname{tr}X \cdot X + \det X \cdot I_2 = O_2 \Rightarrow X^2 = -\det X \cdot I_2 = \mp I_2$

so $X^4 = I_2$

Solution 2: Again letting $X = AB - BA$

Since $X^n = I_2$, X satisfies the polynomial $p(x) = x^n - 1$

the roots of $p(x)$ are $x_k = e^{i2\pi\frac{k}{n}}$, $k \in \{0, 1, \dots, n-1\}$

Since $p(X) = O_2$, the minimal polynomial of X, $m_X(x)$

has the property $m_X(x) \mid p_X(x)$ $(*)$

We also know that $\deg m_X \leq 2$

- if $\deg m_X = 1$, then $(*) \Rightarrow m_X(x) = x - e^{i2\pi\frac{k}{n}}$

and $\lambda = e^{i2\pi\frac{k}{n}}$ is the only eigenvalue of X

but $2\lambda = \operatorname{tr}X = 0$ contradiction.

Hence $\deg m_X = 2$, $(*) \Rightarrow m_X(x) = (x - e^{i2\pi\frac{k_1}{n}})(x - e^{i2\pi\frac{k_2}{n}})$

and $\lambda_1 = e^{i2\pi\frac{k_1}{n}}$, $\lambda_2 = e^{i2\pi\frac{k_2}{n}}$ are the eigenvalues of X

$\lambda_1 + \lambda_2 = \operatorname{tr}X = 0 \Rightarrow e^{i2\pi\frac{k_1}{n}} + e^{i2\pi\frac{k_2}{n}} = 0$

$\lambda_1\lambda_2 = \det X \in \mathbb{R} \Rightarrow e^{i2\pi\frac{k_1}{n}} e^{i2\pi\frac{k_2}{n}} \in \mathbb{R} \Rightarrow e^{i2\pi\frac{k_1+k_2}{n}} \in \mathbb{R} \Rightarrow e^{i2\pi\frac{k_1+k_2}{n}} = \pm 1$

thus $m_X(x) = x^2 - (e^{i2\pi\frac{k_1}{n}} + e^{i2\pi\frac{k_2}{n}})x + e^{i2\pi\frac{k_1}{n}} e^{i2\pi\frac{k_2}{n}} = x^2 \pm 1$

since X satisfies its minimal polynomial

$m_X(X) = O_2 \Rightarrow X^2 \pm I_2 = O_2 \Rightarrow X^4 = I_2$

Solution 3: We can instead try the following simple argument:

Since $\operatorname{tr}X = 0$ let $X = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

then $X^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix} \Rightarrow$

$X^2 = y I_2$, $y = a^2 + bc \in \mathbb{R}$

- if n is odd, $n = 2k+1$

$X^n = I_2 \Rightarrow (X^2)^k \cdot X = I_2 \Rightarrow y^k X = I_2 \Rightarrow$

$\operatorname{tr}(y^k X) = \operatorname{tr} I_2 \Rightarrow y^k \cdot 0 = 2$ contradiction

- if n is even, $n = 2k$

$X^n = I_2 \Rightarrow (X^2)^k = I_2 \Rightarrow y^k I_2 = I_2 \Rightarrow y^k = 1 \xRightarrow{y \in \mathbb{R}} y = \pm 1$

so $X^2 = y I_2 \Rightarrow X^4 = y^2 I_2 = I_2$

Problem 8

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If we simply try multiplying the given equation by A or B (either on the left or on the right) possibly multiple times, we will always get a monomial with a large degree.

We instead try factoring.

$AB = aA + bB \Rightarrow AB - aA - bB + abI_n = abI_n \Rightarrow A(B - aI_n) - b(B - aI_n) = abI_n \Rightarrow$

$(A - bI_n)(B - aI_n) = abI_n \overset{ab \neq 0}{\Longrightarrow} \left(\frac{1}{b}A - I_n\right)\left(\frac{1}{a}B - I_n\right) = I_n \Rightarrow \left(\frac{1}{a}B - I_n\right)^{-1} = \frac{1}{b}A - I_n \Rightarrow$

$\left(\frac{1}{a}B - I_n\right)\left(\frac{1}{b}A - I_n\right) = I_n \Rightarrow (B - aI_n)(A - bI_n) = abI_n \Rightarrow BA - aA - bB + abI_n = abI_n \Rightarrow$

$BA = aA + bB = AB$, A, B commute