Math Competition Preparation Seminar
Discrete Math

December 12, 2024

1 Theory/Background

1.1 Main Ideas

- Invariants(revisited): A number or a property that characterizes a mathematical object. Very useful when dealing with repeatedly applied transformations. Induction is sometimes useful.
- Extremal Principle: Consider an object of a set that maximizes some function.
- Dirichlet's Box Principle: Mapping (at least) $kn+1$ objects to n sets implies that at least one set contains at least k+1 objects. Useful in proving the existence of an object with a specific property.

2 Problems

Problem 1 Consider a rectangular grid of squares, with each square containing a positive integer. In each step you can double all elements of a row or subtract 1 from all elements of a column. Is it possible to attain a grid of zeros for any initial grid?

Problem 2 Is it possible to transform $f(x) = x^2 + 4x + 3$ into $g(x) = x^2 + 10x + 9$ by performing a combination of the following transformations(you can use each transformation multiple times)

$$f(x) \to x^2 f(1/(x+1)) \quad \text{or} \quad f(x) \to (x-1)^2 f(1/(x-1))$$

Problem 3 Consider the set S of 7 vertices of a cube. You can extend this set by reflecting some point X in S with respect to another point Y in S. Can you get the eight vertex of the cube in S?

Problem 4 Let $a, b \in \mathbb{R}$ with $b > a > 0$ and consider the sequences x_n, y_n with $x_0 = a, y_0 = b$ defined recursively as

$$x_{n+1} = \sqrt{x_n y_{n+1}}, \quad y_{n+1} = \sqrt{x_n y_n}$$

Prove that $\lim_{n\to\infty} x_n,\ \lim_{n\to\infty} y_n$ exist and compute them.

Problem 5 Prove that an 8×8 chessboard cannot be covered by 15T-tetrominoes and one square tetromino.

Problem 6 Each element of a 25×25 matrix is either $+1$ or -1. Let a_i be the product of the entries of the i-th row and b_j be the product of the entries of the j-th column. Prove that

$$a_1 + b_1 + .. + a_{25} + b_{25} \neq 0$$

Problem 7 Prove that a 10×10 board cannot be covered by 25 straight tetrominoes(4×1).

Problem 8 Consider $n \geq 4$ lines on the plane in general position. Prove that there exist at least $(2n-2)/3$ triangles whose sides do not intersect with any line and whose interiors are disjoint.

Problem 9 Consider $n \geq 3$ points on the plane. Prove that there exist three points making an angle $\theta \leq \pi/n$.

Problem 10 Prove that every convex polyhedron has at least two faces with the same number of sides.

Problem 11 Solve the following system for $x, y, z \in \mathbb{R}$

$$(x+y)^3 = z, (y+z)^3 = x, (z+x)^3 = y$$

Problem 12 Prove that one of the positive reals $a, 2a, ..., (n-1)a$ has at most distance $1/n$ from a positive integer.

Problem 13 Among $n+1$ distinct integers from $\{1, 2, ..., 2n\}$ there are two which are coprime.

Problem 14 In any convex $2n$-gon, there is a diagonal not parallel to any side.