Problem 1

Πέμπτη, 2 Ιανουαρίου 2025 11:20 πμ

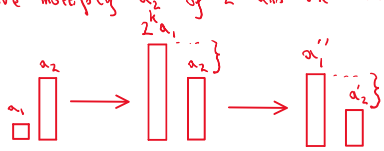
We first choose two squares on the same column and make the equal. We apply this process repeatedly to make all numbers of a column equal and then subtract to attain a column of '0's. We then continue with the next column since the elements of the column of zeros are multiplied by powers of 2, remaining 0 (finger joke).

Idea: Choose two squares on the same column containing a_1, a_2.

No matter how smaller a_1 may be from a_2, we can multiply a_1 by 2 repeatedly until it barely surpasses a_2.

We then subtract 1 repeatedly, so that the difference of the numbers remains constant but a_2 decreases and eventually $a_2' = \frac{a_1'}{2}$.

We multiply a_2' by 2 and the two numbers are equal.



Now let's be precise.

Choose two squares containing $a_1, a_2 : a_1 \neq a_2$. Wlog $a_1 < a_2$

Multiply all the other numbers in the column to ensure that they are larger than a_2.

Choose $k \in \mathbb{N}^* : 2^k \leq \frac{a_2}{a_1} < 2^{k+1}$

Multiply the row of a_1 by 2^{k+1}

Now solving the system of equations $\left.\begin{matrix} 2^{k+1}a_1 - \lambda = 2m \\ a_2 - \lambda = m \end{matrix}\right\}$ we will see if it's possible to subtract '1' λ times and make the integer inside one square half as large as the other integer.

$2^{k+1}a_1 - \lambda = 2m = 2(a_2 - \lambda) \Rightarrow \lambda = 2(a_2 - 2^k a_1) \in \mathbb{N}$ since $a_2 \geq 2^k a_1$

and $m = a_2 - \lambda = 2^{k+1}a_1 - a_2 \in \mathbb{N}^*$ since $a_2 < 2^{k+1}a_1$.

Therefore it is possible to make a_1, a_2 equal. Meanwhile all other numbers in the column remain positive integers since $\lambda < a_2$.

We repeat this process to make all numbers equal and reach a column of '0's. We apply the same for the rest of the columns.

Comment: During our lesson we considered the following claim:

"We can't always make two numbers equal (and thus we can't attain a grid of '0's) because after multiplying the rows of a_1, a_2 by powers of 2 and subtracting we get:

$$\left.\begin{matrix} 2^{k_1}a_1 - l = m \\ 2^{k_2}a_2 - l = m \end{matrix}\right\} \Rightarrow 2^{k_1}a_1 = 2^{k_2}a_2$$

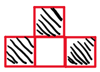
which is not true for all integers a_1, a_2."

This argument is not valid because it overlooks the possibility to multiply and subtract interchangeably. In fact this is exactly what we did in the proof: we multiplied by 2^{k+1}, then subtracted λ and then multiplied by 2 again.

Problem 5(Foivos)

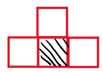
Πέμπτη, 2 Ιανουαρίου 2025 12:19 μμ

Consider that the chessboard has its usual colouring.
Observe that each T-tetromino placed can be coloured in two possible ways:



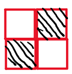
3 white squares
1 black square

or



3 black squares
1 white square

whereas a 2x2 square is coloured uniquely.



2 white squares
2 black squares

Assume there are k T-tetrominoes with 1 white square and l T-tetrominoes with 1 black square. Since $k+l=15$, exactly one of the numbers k,l is even, wlog k: even, l: odd

We now count the total of black squares in two different ways.

$$32 = \underbrace{3k}_{\text{even}} + \underbrace{l}_{\text{odd}} + \underbrace{2}_{\text{even}}$$

The above equation cannot hold true.

Comment: "Counting twice" is a widely used problem solving strategy.

Problem 6(Panos)

Πέμπτη, 2 Ιανουαρίου 2025 12:42 μμ

Let X be the 25×25 matrix and define $P(X)=a_1+b_1+a_2+b_2+\dots+a_{25}+b_{25}$

Given X we start with matrix $Y: y_{ij}=1 \ \forall i,j$

$P(Y)=50$ and change its elements to -1 one by one to eventually get X.

In each step $y_{mn}=1 \to y_{mn}=-1$

and thus $a_m \to -a_m$, $b_n \to -b_n$

while all other a_i, b_j remain constant.

We distinguish 3 distinct cases:

before	$\left.\begin{matrix} a_m=1 \\ b_n=1\end{matrix}\right\}$	$\left.\begin{matrix} a_m=1 \\ b_n=-1\end{matrix}\right\}$	$\left.\begin{matrix} a_m=-1 \\ b_n=-1\end{matrix}\right\}$
	$\downarrow$	$\downarrow$	$\downarrow$
after	$\left.\begin{matrix} a_m=-1 \\ b_n=-1\end{matrix}\right\}$	$\left.\begin{matrix} a_m=-1 \\ b_n=1\end{matrix}\right\}$	$\left.\begin{matrix} a_m=1 \\ b_n=1\end{matrix}\right\}$
$P'-P$	-4	0	4

Therefore after multiple steps $P(X)\equiv P(Y) \pmod 4$

$P(Y)=50 \Rightarrow P(X)\not\equiv 0 \pmod 4 \Rightarrow P(X)\neq 0$

Alternatively,

assume on the contrary that for some X, $P(X)=0$

Consider the sets $A=\{i \mid a_i=-1\}$, $B=\{j \mid b_j=-1\}$

whenever $|A|\geq 2$ ($|A|$ denotes the number of elements of A)

choose $i_1, i_2 \in \{1,2,\dots,25\}: i_1, i_2 \in A$, $i_1\neq i_2$ and exchange all signs in rows i_1, i_2.

$a'_{i_1}=(-1)^{25}a_{i_1}=-a_{i_1}=1$, $a'_{i_2}=1$

After this procedure all b_j remain constant since two signs in each column are exchanged in every step.

Also $P'-P=a'_{i_1}+a'_{i_2}-a_{i_1}-a_{i_2}=1+1-(-1)-(-1)=4$

hence $P'\equiv P \pmod 4$

After applying the above procedure for A, B for as many times as possible we end up with

$|A|=0$ or 1, $|B|=0$ or 1

thus $P(X)\in\{\underbrace{46}_{|A|=|B|=1},\underbrace{48}_{|A|+|B|=1},\underbrace{50}_{|A|=|B|=0}\}$

yet since $P'\equiv P \pmod 4$ in every step, $P(X)=0$ initially, implies $P=48$ in the end, so without loss of generality

$|A|=0$, $|B|=1$ and then

$1=\prod_{i=1}^{25} a_i=\prod_{i,j=1}^{25} x_{ij}=\prod_{j=1}^{25} b_j=-1$ contradiction

Problem 8

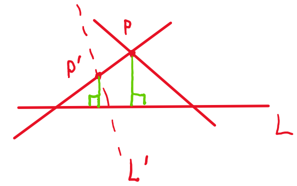
Πέμπτη, 2 Ιανουαρίου 2025 8:47 μμ

The result $\frac{2n-1}{3}$ gives us hints for the proof.
The denominator 3 shows that each triangle is initially counted three times, once for each side.
Additionally, each one of the n lines seems to correspond to 2 triangles, apart from 2 of them.

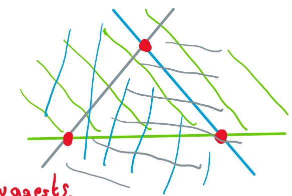
Let S be the set of intersection points of the n lines.

Take line L and one of the half-planes into which it divides the plane. Now choose point P of S that lies on this half-plane, with minimal distance from L.
The (exactly) two lines intersecting at P, also intersect with L (since the n lines are in general position), making a triangle.
The sides of this triangle can't be intersected by another line L', because if this was the case, exactly two sides of the triangle would intersect with L'.

This contradicts the minimality of the distance between P and L as illustrated by the accompanying figure



(This can be proved by a trivial trigonometric observation)
The above can be applied for both half-planes.
Finally, we need to show that there are at most 2 lines such that all points of S lie on only one of the half-planes.
Assume on the contrary, that there are 3 such lines.
The 3 lines intersect two at a time because they are in general position and the points of S all lie inside the triangle as the figure on the right suggests.



A 4th line can only intersect two sides of the triangle, which means that it intersects one of the 3 lines outside the triangle, contradiction!

Problem 10

Παρασκευή, 3 Ιανουαρίου 2025 12:10 πμ

Assume without loss of generality that $x \le y \le z$

$z \le (x+y)^3 \overset{x \le z}{\le} (z+y)^3 = x$, hence $x \le y \le z \le x \Rightarrow x = y = z$

$z = (x+y)^3 \Rightarrow x = 8x^3 \Rightarrow x = 0$ or $x = \pm \frac{1}{2\sqrt{2}}$

$(x, y, z) \in \left\{ (0,0,0), \left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) \right\}$

What would happen if the exponent was 2 instead of 3?