Math Competition Preparation Seminar

Sequences

November 28, 2024

1 Theory/Background

1.1 Main Ideas

- Look for a pattern: When trying to find the general term of a sequence calculate the first few terms and look for a pattern. Make your guess and prove it by induction.
- Search for recursive relations, linear recursions are easy to solve.

1.2 Theorem

The general term of a sequence satisfying the linear recurrence relation:

$$x_{n+k} + a_{k-1}x_{n+k-1} + .. + a_0 x_n = 0, \forall n \in \mathbb{N}_0$$

is given by:

$$x_n = b_1 z_1^n + b_2 z_2^n + .. + b_k z_k^n$$

where $b_1, .., b_k$ are complex numbers to be determined by the initial conditions and $z_1, .., z_k$ are the solutions of the "characteristic equation"($z_1, .., z_k$ are assumed to be distinct):

$$z^k + a_{k-1}z^{k-1} + ..a_0 = 0$$

1.3 Theorem(Weierstrass)

An increasing(resp. decreasing) sequence of real numbers that is bounded above(resp. bellow) converges.

2 Problems

Problem 1 Find a formula for the general term of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, ...$$

Problem 2 The sequence a_n of real numbers satisfies the relation

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n}), \forall m, n \in \mathbb{N}_0$$

If $a_1 = 1$, find a_n.

Problem 3 Find the general term y_n of the sequence of differentiable functions, with $y_0(x) = 1$, satisfying the following recurrence relation

$$y_{n+1}(x) = \frac{d}{dx}(y_n(x) sin^2 x), \forall n \in \mathbb{N}$$

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Problem 4 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2001, \forall n \in \mathbb{N}$$

Problem 5 Find a closed formula for the general term of the sequence defined recursively by $x_1 = 1, x_n = x_{n-1} + n$ whenever n is odd, $x_n = x_{n-1} + n - 1$ whenever n is even.

Problem 6 Let A and E be opposite vertices of a regular octagon. A frog starts at vertex A. From any vertex of the octagon except E, it may jump to one of its two adjacent vertices. Once the frog reaches E, it stops jumping. Let a_n denote the number of distinct paths from A to E of n total jumps. Find the formula of a_n in terms of n.

Problem 7 Assume that $a_1 \geq a_2 \geq .. \geq a_n \geq ... \geq 0$ and the sequence $s_n = \sum_{k=1}^{n} \epsilon_k a_k$ is convergent, where $\epsilon_k = \pm 1$. Prove that

$$\lim_{n\to\infty} (\epsilon_1 + .. + \epsilon_k) a_n = 0$$

Problem 8 Let c, x_0 be positive. Define the sequence

$$x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n}), n \in \mathbb{N}_0$$

Find the limit of the sequence if it exists.

Problem 9 Prove that the following sequence is convergent.

$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{... + \sqrt{n}}}}$$

Problem 10 Let a_n be a sequence of real numbers defined recursively by

$$a_{n+1} = \sqrt{a_n^2 + a_n - 1}, n \in \mathbb{N}$$

Prove that $a_1 \notin (-2, 1)$.