Problem 1(Konstantinos)

Τρίτη, 31 Δεκεμβρίου 2024 11:22 πμ

Firstly, since '1' appears 1 time, '2' appears 2 times, .., 'n' appears n times, we conclude that the last term equal to n is $a_{1+2+..+n} = a_{\frac{n(n+1)}{2}}$. In fact $a_m = n \Leftrightarrow m \in \left\{ \frac{(n-1)n}{2}+1, \frac{(n-1)n}{2}+2, .., \frac{n(n+1)}{2} \right\}$ (*)

Notice $a_1 = 1$

$a_{1+1} = 2$, $a_{1+2} = 2$

$a_{1+2+1} = 3$, $a_{1+2+2} = 3$, $a_{1+2+3} = 3$

$a_{1+2+3+1} = a_{1+2+3+2} = a_{1+2+3+3} = a_{1+2+3+4} = 4$

We will prove that $n = 1+2+..+a_n - 1 + k$, $k \in \{1, 2, .., a_n\}$

We see that the above relation holds for a_1 (trivially).

• If $k \in \{1, 2, .., a_n - 1\}$, then $n = 1+2+..+a_n - 1 + k$

and $n+1 = 1+2+..+a_n - 1 + (k+1)$ $(k+1) \in \{2, 3, .., a_n\}$ (1)

but since $\underbrace{1+2+..+a_n-1+1}_{\frac{(a_n-1)a_n}{2}+1} \leq n < n+1 \leq \underbrace{1+2+..+a_n}_{\frac{a_n(a_n+1)}{2}}$

(*) implies $a_n = a_{n+1}$ hence

$(1) \Rightarrow n+1 = 1+2+..+a_{n+1} - 1 + (k+1)$ $(k+1) \in \{2, 3, .., a_{n+1}\}$

• If $k = a_n$, then $n = 1+2+..+a_n - 1 + a_n$ (2)

hence $\frac{a_n(a_n+1)}{2} = n < n+1 \leq \frac{(a_n+1)(a_n+2)}{2}$ meaning $a_{n+1} = a_n + 1$

so $(2) \Rightarrow n+1 = 1+2+..a_{n+1} - 1 + k$, $k = 1$

Therefore by induction $n = 1+2+..+a_n - 1 + k$, $k \in \{1, 2, .., a_n\}$

Now $1 \leq k \leq a_n \Rightarrow 1+2+..+a_n - 1 + 1 \leq 1+2+..+a_n - 1 + k \leq 1+2+..+a_{n-1} + a_n \Rightarrow$

$\frac{(a_n-1)a_n}{2} + 1 \leq n \leq \frac{a_n(a_n+1)}{2}$

• $n \leq \frac{a_n(a_n+1)}{2} \Rightarrow a_n^2 + a_n - 2n \geq 0$, $\Delta = 1+8n$,

$\left(a_n - \frac{-1+\sqrt{1+8n}}{2}\right)\left(a_n - \frac{-1-\sqrt{1+8n}}{2}\right) \geq 0 \overset{a_n > 0}{\Longrightarrow} a_n \geq \frac{-1+\sqrt{1+8n}}{2}$

• $n \geq \frac{(a_n-1)a_n}{2} + 1 \Rightarrow a_n^2 - a_n - 2(n-1) \leq 0 \Rightarrow a_n \leq \frac{1+\sqrt{1+8(n-1)}}{2}$

notice that $\frac{1+\sqrt{1+8(n-1)}}{2} - \frac{-1+\sqrt{1+8n}}{2} = 1 - \frac{\sqrt{1+8n} - \sqrt{1+8(n-1)}}{2} =$

$1 - \frac{4}{\sqrt{1+8n} + \sqrt{1+8(n-1)}} \in [0, 1)$

this means that there is at most one integer between the upper and the lower bound.

$\frac{1+\sqrt{1+8(n-1)}}{2} \geq a_n \geq \frac{-1+\sqrt{1+8n}}{2} \overset{a_n \in \mathbb{N}}{\Longrightarrow} a_n = \left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil$

Alternatively,

define $f: \mathbb{R}_+ \to \mathbb{R}_+ : f(x) = \frac{x(x+1)}{2}$, $\forall x \in \mathbb{R}_+$

f is strictly increasing and onto, so $\exists f^{-1}$

$f(x) = \frac{x(x+1)}{2} \Rightarrow x^2 + x - 2f(x) = 0$ $\Delta = 1 + 8f(x)$, $x = \frac{-1 \pm \sqrt{1+8f(x)}}{2}$

since $x > 0$, $f^{-1}(y) = \frac{-1+\sqrt{1+8y}}{2}$, $\forall y > 0$

Now $a_m = n \Leftrightarrow \frac{(n-1)n}{2} < m \leq \frac{n(n+1)}{2} \Leftrightarrow f(n-1) < m \leq f(n) \overset{f^{-1}\nearrow}{\Longleftrightarrow} n-1 < f^{-1}(m) \leq n$

Therefore $a_m = \lceil f^{-1}(m) \rceil = \left\lceil \frac{-1+\sqrt{1+8m}}{2} \right\rceil$

Problem 3

Κυριακή, 29 Δεκεμβρίου 2024 6:49 μμ

First calculate a few terms

$y_0(x)=1$

$y_1(x)=\frac{d}{dx}\left(y_0(x)\sin^2 x\right)=\frac{d}{dx}(\sin^2 x)=2\sin x\cos x=\sin 2x$

$y_2(x)=\frac{d}{dx}\left(y_1(x)\sin^2 x\right)=\frac{d}{dx}\left(\sin 2x\sin^2 x\right)=2\cos 2x\sin^2 x+\sin 2x\,2\sin x\cos x=$

$2\sin x(\cos 2x\sin x+\sin 2x\cos x)=2\sin x\sin 3x$

$y_3(x)=\frac{d}{dx}\left(2\sin x\sin 3x\sin^2 x\right)=2\left(3\cos 3x\sin^3 x+\sin 3x\cdot 3\sin^2 x\cdot\cos x\right)=$

$6\sin^2 x(\cos 3x\sin x+\sin 3x\cos x)=6\sin^2 x\sin 4x$

At this point we guess that $y_n(x)=n!\sin^{n-1}x\sin(n+1)x$

Let's prove it by induction

Assuming $y_n(x)=n!\sin^{n-1}x\sin(n+1)x$

$y_{n+1}(x)=\frac{d}{dx}\left(n!\sin^{n-1}x\sin(n+1)x\cdot\sin^2 x\right)=n!\frac{d}{dx}\left(\sin(n+1)x\cdot\sin^{n+1}x\right)=$

$n!\left[(n+1)\cos(n+1)x\sin^{n+1}x+\sin(n+1)x\cdot(n+1)\sin^n x\cos x\right]=$

$n!(n+1)\sin^n x\left[\cos(n+1)x\sin x+\sin(n+1)x\cos x\right]=(n+1)!\sin^n x\sin(n+2)x$ ✓

Our guess is correct.

Problem 5

Κυριακή, 29 Δεκεμβρίου 2024 7:03 μμ

Let's calculate a few terms

odd	even
$x_n = x_{n-1} + n$	$x_n = x_{n-1} + n - 1$
$x_1 = 1$	$x_2 = 2$
$x_3 = 5$	$x_4 = 8$
$x_5 = 13$	$x_6 = 18$
$x_7 = 25$	$x_8 = 32$

Whenever n is even, we add the same number twice to get to the next even number.

Let's be precise

$$x_{2n+2} = x_{2n+1} + 2n+1 = x_{2n} + 2n+1 + 2n+1 \Rightarrow x_{2n+2} = x_{2n} + 4n+2$$

Applying this equality repeatedly

$$x_{2n} = x_{2n-2} + 4(n-1) + 2 = x_{2n-4} + 4(n-2) + 2 + 4(n-1) + 2 = \dots =$$

$$x_2 + \underbrace{4(n-1) + 2 + 4(n-2) + 2 + \dots + 4 + 2}_{2(n-1)\text{ terms}} = x_2 + \sum_{k=1}^{n-1} 4k + \sum_{k=1}^{n-1} 2 =$$

$$x_2 + 4\sum_{k=1}^{n} k + 2\sum_{k=1}^{n-1} 1 = 2 + 4\frac{(n-1)n}{2} + 2(n-1) = 2n^2$$

also $x_{2n+1} = x_{2n} + 2n+1 = 2n^2 + 2n + 1$

Therefore $x_n = \begin{cases} 2\left(\frac{n}{2}\right)^2 & n\text{: even} \\ 2\left(\frac{n-1}{2}\right)^2 + 2\frac{n-1}{2} + 1 & n\text{: odd} \end{cases} \Rightarrow$

$$x_n = \begin{cases} \frac{n^2}{2}, & n\text{: even} \\ \frac{n^2+1}{2}, & n\text{: odd} \end{cases}$$

Problem 7

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Firstly, if we were to instantly apply the triangle inequality then:

$$|(\varepsilon_1+\varepsilon_2+\dots+\varepsilon_n)a_n| \le (|\varepsilon_1|+|\varepsilon_2|+\dots+|\varepsilon_n|)a_n \overset{|\varepsilon_k|=1}{=\!=} n a_n$$

This is like supposing that all ε_k are 1 even though we are only given that $\sum_{k=1}^{\infty} \varepsilon_k a_k$ converges (instead of $\sum_{k=1}^{\infty} a_k$ converges, a_k may converge to 0 "slowly".

We will first work with the sum of the ε_k's.

$$\sum_{k=m}^{n} \varepsilon_k \overset{a_k\neq 0}{=} \sum_{k=m}^{n} \frac{\varepsilon_k a_k}{a_k} = \sum_{k=m}^{n} \frac{\sum_{j=1}^{k} \varepsilon_j a_j - \sum_{j=1}^{k-1} \varepsilon_j a_j}{a_k} =$$

$$\sum_{k=m}^{n} \frac{s_k - s_{k-1}}{a_k} = \sum_{k=m}^{n} \frac{s_k}{a_k} - \sum_{k=m}^{n} \frac{s_{k-1}}{a_k} =$$

$$\frac{s_n}{a_n} + \sum_{k=m}^{n-1} \frac{s_k}{a_k} - \sum_{k=m}^{n-1} \frac{s_k}{a_{k+1}} - \frac{s_{m-1}}{a_m} =$$

$$\sum_{k=m}^{n-1} s_k\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) + \frac{s_n}{a_n} - \frac{s_{m-1}}{a_m} =$$

$$\sum_{k=m}^{n-1} (s_k - s_n)\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) + \sum_{k=m}^{n-1} s_n\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) + \frac{s_n}{a_n} - \frac{s_{m-1}}{a_m} =$$

$$\sum_{k=m}^{n-1} (s_k - s_n)\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) + s_n\left(\frac{1}{a_m} - \cancel{\frac{1}{a_{m+1}}} + \cancel{\frac{1}{a_{m+1}}} - \cancel{\frac{1}{a_{m+2}}} + \dots + \cancel{\frac{1}{a_{n-1}}} - \frac{1}{a_n}\right) +$$

$$+ \frac{s_n}{a_n} - \frac{s_{m-1}}{a_m} =$$

$$\sum_{k=m}^{n-1} (s_k - s_n)\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) + s_n\left(\frac{1}{a_m} - \cancel{\frac{1}{a_n}}\right) + \cancel{\frac{s_n}{a_n}} - \frac{s_{m-1}}{a_m} =$$

$$\sum_{k=m}^{n-1} (s_k - s_n)\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) + \frac{s_n - s_m}{a_m}$$

At this point we can utilize the convergence of s_n and in fact use the same trick with the telescopic sum that we used in the previous calculation.

s_n: converges $\Rightarrow s_n$: Cauchy sequence

Keep in mind that s_n: converges $\to |s_n| \le M$ and $a_n \to 0$

Take $\varepsilon > 0$ and $m \in \mathbb{N}: |s_n - s_k| < \varepsilon \quad \forall n,k \ge m$

$$|a_n \sum_{k=1}^{n} \varepsilon_k| \le a_n\left(\left|\sum_{k=1}^{m-1} \varepsilon_k\right| + \left|\sum_{k=m}^{n} \varepsilon_k\right|\right) \le a_n\left(m-1 + \left|\sum_{k=m}^{n} \varepsilon_k\right|\right) =$$

$$a_n\left(m-1 + \left|\sum_{k=m}^{n-1} (s_k - s_n)\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) + \frac{s_n - s_m}{a_m}\right|\right) \le$$

$$a_n\left(m-1 + \sum_{k=m}^{n-1} |s_k - s_n| \left|\frac{1}{a_k} - \frac{1}{a_{k+1}}\right| + \frac{|s_n - s_m|}{a_m}\right) \overset{a_n \ge a_{n+1}}{<}$$

$$a_n\left(m-1 + \sum_{k=m}^{n-1} \left(\frac{1}{a_{k+1}} - \frac{1}{a_k}\right)\varepsilon + \frac{2M}{a_m}\right)$$

$$a_n\left[m-1 + \left(\frac{1}{a_n} - \frac{1}{a_m}\right)\varepsilon + \frac{|s_n - s_m|}{a_m}\right] =$$

$$a_n\left[m-1 - \frac{\varepsilon}{a_m} + \frac{2M}{a_m}\right] + \varepsilon \xrightarrow{n\to\infty} \varepsilon$$

Finally $\limsup_{n\to\infty} |a_n \sum_{k=1}^{n} \varepsilon_k| < \varepsilon \quad \forall \varepsilon > 0$

hence $\lim_{n\to\infty} a_n \sum_{k=1}^{n} \varepsilon_k = 0$

Comment: Notice that we applied the triangle inequality for the first m terms of the sum (instead for all of them) and then let n tend to infinity.

This proof was given by prof. Sakellaris from Thessaloniki.