Problem 2

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Let $A_1, A_2, \ldots, A_6$ be the 6 colours and the cube with its faces painted with them. Consider the face with colour A_1. There are 5 choices for the colour of the opposite face. Without loss of generality assume the opposite face is painted with A_2. Now consider the face with colour A_3. There are 3 choices for the colour of the opposite face. Finally there are 2 possible arrangments for the 2 remaining faces. Thus the answer is: $5 \cdot 3 \cdot 2 = 30$.

Problem 3

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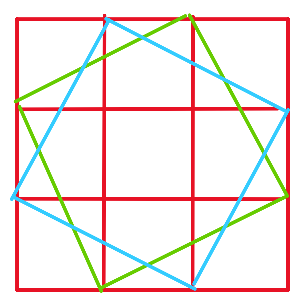
i) Squares with side length i can have their lower left corner at any (x,y) with $0 \le x \le n-i$, $0 \le y \le n-i$ to be contained entirely in the grid. Hence there are $(n-i+1)^2$ such squares.

Summing over all squares, the answer is

$$\sum_{i=1}^{n} (n-i+1)^2 \;\overline{\overline{\;\scriptstyle 1 \le i \le n\;}}_{1 \le j = n-i-1 \le n}\; \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

ii) We will try to use the previous result. Define $\{0,1,\dots,n\}^4 \to \mathbb{N}$ $d_1(x,y,x_0,y_0) = |x-x_0| + |y-y_0|$ and consider the squares we previously counted. Inside a square of side length i with sides parallel to the coordinate axes, more squares can be inscribed. The sides of these squares have the same "1-length", $d_1 = i$. There are i such squares counting the initial square itself. The answer now is



$$\sum_{i=1}^{n} \underbrace{(n-i+1)^2}_{\text{squares with sides parallel to axes}} \cdot \underbrace{i}_{\text{inscribed}} = \sum_{j=1}^{n} j^2 (n-j+1) = (n+1)\sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} j^3 =$$

$$(n+1)\frac{n(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2 = n(n+1)^2\left(\frac{2n+1}{6} - \frac{n+1}{4}\right) =$$

$$n(n+1)^2 \frac{2n-2}{12} = \frac{(n-1)n(n+1)^2}{6}$$

Problem 4

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Consider the function $F(z)=\sum_{n=0}^{\infty} x_n z^n$.

$$F(z)=\sum_{n=0}^{\infty} x_n z^n = x_0+\sum_{n=1}^{\infty} x_n z^n = x_0+\sum_{n=1}^{\infty}(a x_{n-1}+b^n)z^n = x_0 + a\sum_{n=1}^{\infty} x_{n-1}z^n+\sum_{n=1}^{\infty} b^n z^n =$$

$$x_0+a\sum_{n=0}^{\infty} x_n z^{n+1} + \sum_{n=1}^{\infty} b^n z^n = x_0 + azF(z)+\frac{bz}{1-bz} \Rightarrow (1-az)F(z)=1+\frac{bz}{1-bz} \Rightarrow$$

$$F(z)=\frac{1}{(1-az)(1-bz)}$$

<u>Case 1: $a\neq b$</u> Write $F(z)=\frac{1}{(1-az)(1-bz)}=\frac{A}{1-az}+\frac{B}{1-bz}$, then

$$A=\frac{1}{1-bz}\Big|_{z=\frac{1}{a}}=\frac{1}{1-\frac{b}{a}}=\frac{a}{a-b}, \quad B=\frac{1}{1-az}\Big|_{z=\frac{1}{b}}=\frac{1}{1-\frac{a}{b}}=\frac{b}{b-a}$$

Now $\sum_{n=0}^{\infty} x_n z^n = F(z) = \frac{A}{1-az}+\frac{B}{1-bz} = A\sum_{n=0}^{\infty} a^n z^n + B\sum_{n=0}^{\infty} b^n z^n \Rightarrow$

$$x_n = Aa^n+Bb^n=\frac{a}{a-b}a^n+\frac{b}{b-a}b^n=\frac{a^{n+1}-b^{n+1}}{a-b} \qquad n\in\mathbb{N}$$

<u>Case 2: $a=b$</u> $\sum_{n=0}^{\infty} x_n z^n = F(z)=\frac{1}{(1-az)^2}=\frac{1}{a}\frac{d}{dz}\left(\frac{1}{1-az}\right)=\frac{1}{a}\frac{d}{dz}\sum_{n=0}^{\infty} a^n z^n =$

$$\frac{1}{a}\sum_{n=0}^{\infty} a^n n z^{n-1} = \frac{1}{a}\sum_{n=1}^{\infty} a^n n z^{n-1} = \frac{1}{a}\sum_{n=0}^{\infty} a^{n+1}(n+1)z^n$$

hence $x_n=\frac{1}{a}a^{n+1}(n+1)=(n+1)a^n$, $n\in\mathbb{N}$

Problem 5

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Normally we would approach this problem as follows

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \Rightarrow \frac{d}{dx}(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx} x^k \Rightarrow n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \cdot k x^{k-1}$$

$$\overset{x=1}{\Longrightarrow} n \cdot 2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}$$

We can try a combinatorial argument instead

- We think $\binom{n}{k}$ as the number of subsets of $\{1, 2, \dots, n\}$ consisting of k elements. Then we can choose one of these k elements in k ways.
- Consider $S_k = \{(T, x) : T \subseteq \{1, \dots, n\}, |T| = k, x \in T\}$, then as explained above $|S_k| = k\binom{n}{k}$
- Now consider $S' = \{(T, x) : x \in \{1, \dots, n\}, T \subseteq \{1, \dots, n\}, x \in T\}$, $|S'| = n 2^{n-1}$ because there are n choices for x and 2^{n-1} subsets of $\{1, \dots, n\}$ containing x.
- Lastly we consider $f : \bigcup_{k=0}^{n} S_k \to S'$, with $(T, x) \mapsto (T, x)$

If $\sigma_1 = (T_1, x_1), \sigma_2 = (T_2, x_2) \in \bigcup_{k=0}^{n} S_k : f(\sigma_1) = f(\sigma_2) \Rightarrow (T_1, x_1) = (T_2, x_2) \Rightarrow \sigma_1 = \sigma_2$ hence f is injective

If $(T_0, x_0) \in S'$, then $T_0 \subseteq \{1, \dots, n\} \Rightarrow 0 \leq |T| \leq n$, $x_0 \in T_0$

thus $(T_0, x_0) \in S_{|T_0|}$ and $f((T_0, x_0)) = (T_0, x_0)$ $\forall T_0, x_0$ hence f is surjective

- Therefore f is bijective and since S_k are disjoint

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n} |S_k| = \left|\bigcup_{k=0}^{n} S_k\right| = |S'| = n 2^{n-1}$$

Problem 9

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- Let $p(n)$ denote the number of partitions of n.

Then the relation $G_p(x) = \sum_{n=0}^{\infty} p(n)x^n = (1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\dots$ holds.

Take a partition λ of n consists of a_1 '1's, a_2 '2's, ..., a_n 'n's. Then it corresponds to choosing x^{a_1} from the first parenthesis, choosing $x^{a_2 \cdot 2}$ from the second parenthesis, etc and multiplying them to get x^n. Thus x^n occurs $p(n)$ times.

- Similar reasoning yields expressions for the generating functions for the number of partitions into odd integers $o(n)$ and the number of partitions into distinct integers.

In fact

- $G_d(x) = \sum_{n=0}^{\infty} d(n)x^n = (1+x)(1+x^2)\dots = \prod_{k=0}^{\infty}(1+x^k)$

(Notice that a partition may contain an integer 0 or 1 time which leads to choosing 0 or 1 respectively from the corresponding parentheses).

- $G_o(x) = \sum_{n=0}^{\infty} o(n)x^n = (1+x+x^2+\dots)(1+x^3+x^6+x^9+\dots)\dots = \prod_{k=0}^{\infty}(1+x^{2k+1}+x^{2(2k+1)}+\dots)$

(Notice that we only kept parentheses corresponding to odd numbers)

- Finally $G_o(x) = \prod_{k=0}^{\infty}(1+x^{2k+1}+x^{2(2k+1)}+\dots) = \prod_{k=0}^{\infty}\frac{1}{1-x^{2k+1}} = \frac{1}{\prod_{k=0}^{\infty}(1-x^{2k+1})} = \frac{\prod_{k=0}^{\infty}(1-x^{2k})}{\prod_{k=0}^{\infty}(1-x^{2k+1})\prod_{k=0}^{\infty}(1-x^{2k})} =$

$$\frac{\prod_{k=0}^{\infty}(1-x^{2k})}{\prod_{k=0}^{\infty}(1-x^k)} = \frac{\prod_{k=0}^{\infty}(1-x^k)(1+x^k)}{\prod_{k=0}^{\infty}(1-x^k)} = \prod_{k=0}^{\infty}(1+x^k) = G_d(x)$$

Therefore $G_o(x) = G_d(x) = \sum_{n=0}^{\infty} o(n)x^n = \sum_{n=0}^{\infty} d(n)x^n \Rightarrow o(n) = d(n) \quad \forall n \in \mathbb{N}$