Math Competition Preparation Seminar

Counting and generating functions

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1 Theory/Background

1.1 Definition(Bijection)

Let A, B be two sets and $f : A \to B$. f is a bijection between A and B if

- $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$, $\forall x_1, x_2 \in A$.
- $\forall b \in B, \exists a \in A$, such that $f(a) = b$.

1.2 Definition(Partition)

A finite sequence of integers λ is called a partition of a positive integer n if $\lambda = (\lambda_1, \lambda_2, \lambda_k)$ such that $\lambda_1 + .. + \lambda_k = n$ and $\lambda_1 \leq \lambda_2 \leq .. \leq \lambda_k$.

1.3 Definition(Generating function)

The ordinary generating function of a sequence $a_{n n\in\mathbb{N}}$ is the function given by $G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n$.

1.4 Theorem

Let A, B be finite sets. Then

$$\exists f : A \to B, f : \text{bijection} \iff |A| = |B|.$$

where $|S|$ denotes the cardinality of the set S.

2 Problems

Problem 1 Compute the number of solutions of the following equation for $m, n \in \mathbb{N}^*$

$$x_1 + x_2 + .. + x_m = n \;,\; x_i \in \mathbb{N}, \forall i \in \{1, 2, .., m\}$$

Problem 2 In how many distinct ways can the faces of a cube be painted using 6 different colours, such that each face has a different colour?

Problem 3 Let $n \in \mathbb{N}^*$. Compute the number of squares with vertices belonging to the set $\{(x, y) \in \mathbb{R} : x, y \in \{0, 1, .., n\}\}$ and with their sides parallel to the x or y axis. What if their sides aren't necessarily parallel to the coordinate axis?

Problem 4 Find the general term of the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by

$$x_n = ax_{n-1} + b^n,\ \forall n \geq 1,\ x_0 = 1$$

where $a, b \in \mathbb{R}$.

Problem 5 Compute the sum $\sum_{k=0}^{n} k\binom{n}{k}$. Try using a combinatorial argument.

Problem 6 How many subsets of $\{1, 2, .., n\}$ have no two consecutive numbers.

Problem 7 In how many ways can you triangulate a convex n-gon?

Problem 8 Prove that the Fibonacci numbers satisfy the following

$$f_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2}.$$

Problem 9 Prove that the number of partitions of a positive integer n into odd integers is equal to the number of partitions into distinct integers.

Problem 10 Consider n seats with exactly 1 person sitting on each one. Each person is allowed to move to a neighbouring seat. In how many ways can the n people rearrange themselves?