Math Competition Preparation Seminar
Methods of Proof

November 21, 2024

1 Theory/Background

1.1 Main Ideas

- Argument by contradiction: After assuming that a statement does not hold, contradict your hypothesis or deduce that a fact known to be true is false.
- Induction: Whenever a property $P(n)$ is true for some integer n_0 and $P(k)$ is true implies $P(k+1)$ is true $\forall\, k \geq n_0$ we conclude that $P(n)$ holds $\forall\, n \geq n_0$. You may also use more elaborate forms of induction.
- Invariants: A number or a property that characterizes a mathematical object. Very useful when dealing with repeatedly applied transformations.
- Extremal Principle: Assuming some ordering of a finite set, you may choose an element that is maximal or minimal.

2 Problems

Problem 1 Examine whether $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is rational or not.

Problem 2 Paint all points of $\mathbb{R}^3$ red, blue or green. Prove that for some colour, the set S of points painted with this colour has the following property:
For all $d > 0$ there exist two points in S lying at a distance d from one another.

Problem 3 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$nf(m) + mf(n) = (n+m)f(n^2+m^2)$$

for all $n, m \in \mathbb{N}$.

Problem 4 Prove that the interval $[0, 1]$ cannot be partitioned into two disjoint sets A and B such that $B = A + a$ for some $a \in \mathbb{R}$.

Problem 5 Prove that

$$n^{n+1} > (n+1)^n$$

for all $n \geq 3$.

Problem 6 Show that an isosceles triangle with one angle equal to 120^o can be dissected to n triangles similar to the original one, where $n \geq 4$.

Problem 7 Let N be a positive integer. Prove that

$$\sum_{n=1}^{N} \frac{1}{n^3} \leq \frac{3}{2}$$

Problem 8 Prove the Arithmetic mean - Geometric mean (AM-GM) inequality for all $a_1, a_2, .., a_n > 0, n \in \mathbb{N}$:

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n}$$

Problem 9 In the next figure you may change the sign of all elements in a row, a column or a parallel to one of the diagonals. Is it possible to attain a grid full of '1's?

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

Problem 10 Solve the following equation for the real number x:

$$(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x$$

Problem 11 The number 99...99 (1997 '9's) is initially written on the blackboard. In each step you take one number written on the blackboard, factor it into two factors and erase it, then each factor is (independently) increased or decreased by 2 and the resulting two numbers are written on the board. Is it possible to make all numbers on the blackboard equal to 9?

Problem 12 Prove that if n points of the plane do not lie on the same line, then there exists a line passing through exactly two points.