Problem 1

Πέμπτη, 26 Δεκεμβρίου 2024 10:20 πμ

Let $r = \sqrt{2} + \sqrt{3} + \sqrt{5}$

The only thing we can do here is square both sides to decrease the number of square roots in the equation

$r = \sqrt{2} + \sqrt{3} + \sqrt{5} \Rightarrow r - \sqrt{5} = \sqrt{2} + \sqrt{3} \Rightarrow (r - \sqrt{5})^2 = (\sqrt{2} + \sqrt{3})^2 \Rightarrow r^2 - 2r\sqrt{5} + 5 = 2 + 3 + 2\sqrt{6} \Rightarrow$

$r^2 = 2r\sqrt{5} + 2\sqrt{6}$

Again

$r^4 = (2r\sqrt{5} + 2\sqrt{6})^2 = 20r^2 + 4r\sqrt{30} + 72$

Solving for $\sqrt{30}$: $\sqrt{30} = \dfrac{r^4 - 20r^2 - 72}{4r}$

If we now assume that r is rational, we can write $r = \frac{p}{q}$, $p, q \in \mathbb{Z}$ $\gcd(p,q) = 1$ with $p, q \neq 0$. Substituting yields:

$$\sqrt{30} = \frac{p^4 - 20p^2q^2 - 72q^4}{4pq^3}$$

both the numerator and the denominator are integers, hence $\sqrt{30}$ is rational

Write $\sqrt{30} = \frac{\alpha}{\beta}$, $\alpha, \beta \in \mathbb{N}$, $\underline{\gcd(\alpha, \beta) = 1}$

$\alpha = \sqrt{30}\beta \Rightarrow \alpha^2 = 30\beta^2$

Now $5 \mid 30\beta^2 \Rightarrow 5 \mid \alpha^2 \Rightarrow 5 \mid \alpha$ so $\alpha = 5k$

$\alpha^2 = 30\beta^2 \Rightarrow 5k^2 = 6\beta^2$

$5 \mid 5k^2 \Rightarrow 5 \mid 6\beta^2 \Rightarrow 5 \mid \beta^2 \Rightarrow 5 \mid \beta$, so $5 \mid \alpha$, $5 \mid \beta$, but $\gcd(\alpha, \beta) = 1$ contradiction

$\underline{r \text{ is irrational}}$

Problem 2

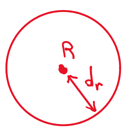
Πέμπτη, 26 Δεκεμβρίου 2024 10:46 πμ

Assume on the contrary that for each colour there is some positive real number, that is not the distance of any pair of points of the same colour. Denote these numbers as d_r, d_b, d_g for red, blue and green respectively.

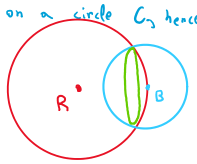
Without loss of generality assume $d_r \geq d_b \geq d_g$

We will only consider the case in which points of all colours exist.

Choose a red point R in $\mathbb{R}^3$ and notice that the points on the sphere with center R and radius d_r can only be blue or green

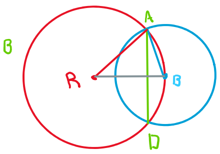


Choose a blue point B on the sphere and consider the sphere with center B and radius d_b. The points of this sphere can only be red or green. The new sphere also intersects the one with the red center (because $d_b < d_r$) on a circle C, hence the points of this circle can only be green.



We now prove that the diameter of this circle is greater than d_g

$AR = BR = d_r$, $AB = d_b$



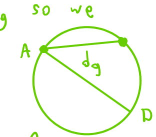
Since AR, BR are longer than (or as long as) AB they correspond to the largest angles of $\overset{\triangle}{ABR}$ meaning $A\hat{B}R = B\hat{A}R \geq A\hat{R}B$

hence $A\hat{B}R \geq 60°$

Now considering the triangle $\overset{\triangle}{ABD}$, $A\hat{B}D \geq 120°$

hence AD is the longest side, $AD \geq AB = d_b \geq d_g$

Finally the diameter of circle C is langer than d_g so we can find 2 green points lying on C with distance equal to d_g. Contradiction!



(The cases left out, namely the ones that some colour isn't used and the case when the red-center sphere has no blue points are easily checked.)

Comment: We used the ordering of the numbers d_r, d_b, d_g to ensure that

1) the two spheres intersect,
2) We can find a chord of C of length d_g.

Problem 4

Πέμπτη, 26 Δεκεμβρίου 2024 4:56 μμ

- First, A, B: partition $[0,1]$ means

  $A \cup B = [0,1]$, $A \cap B = \emptyset$

  Also, $B = A + \alpha \Leftrightarrow B = \{a + \alpha \mid a \in A\} \Leftrightarrow A = B - \alpha$

  So we may assume without loss of generality that $\alpha > 0$

- Suppose $\exists x \in [0, \alpha) : x \in B$

  then $x \in B \Rightarrow x \in (A + \alpha) \Rightarrow (x - \alpha) \in A$

  but $x < \alpha \Rightarrow x - \alpha < 0$ contradiction, because $A \subseteq [0,1]$

  hence $[0, \alpha) \subseteq A$

- Now $[0, \alpha) \subseteq A \xRightarrow{B = A + \alpha} [\alpha, 2\alpha) \subseteq B$

  We will continue in this fashion until we surpass 1 but we need to be careful.

  Choose $k \in \mathbb{N} : 2k\alpha \leq 1 < 2(k+1)\alpha$

Induction hypothesis: $\exists n \in \mathbb{N}: [0,\alpha) \subseteq A, [\alpha, 2\alpha) \subseteq B, [2\alpha, 3\alpha) \subseteq A, \dots,$ $[(2n-2)\alpha, (2n-1)\alpha) \subseteq A$, $[(2n-1)\alpha, 2n\alpha) \subseteq B$ and $n \leq k-1$

- Suppose that $\exists x \in [2n\alpha, (2n+1)\alpha) : x \in B$ (notice $(2n+1)\alpha \leq (2k-1)\alpha < 1$)

  $x \in B \xRightarrow{B = A + \alpha} (x - \alpha) \in A$, but $2n\alpha \leq x < (2n+1)\alpha \Rightarrow (2n-1)\alpha \leq x - \alpha < 2n\alpha$

  the above leads to a contradiction because $[(2n-1)\alpha, 2n\alpha) \subseteq B$

  hence $[2n\alpha, (2n+1)\alpha) \subseteq A$

- $[2n\alpha, (2n+1)\alpha) \subseteq A \xRightarrow{B = A + \alpha} [(2n+1)\alpha, (2n+2)\alpha) \subseteq B$ (notice $(2n+2)\alpha \leq 2k\alpha \leq 1$)

(The above induction can be omitted if $k = 1$)

Therefore the abs induction yields $[0, \alpha) \subseteq A$, $[\alpha, 2\alpha) \subseteq B, \dots,$ $[(2k-2)\alpha, (2k-1)\alpha$ A, $[(2k-1)\alpha, 2k\alpha) \subseteq B$

We now consider 2 cases:

- Case 1: $2k\alpha \leq 1 < (2k+1)\alpha$

  we notice $1 \in A \Rightarrow 1 + \alpha \in B$ impossible because $B \subseteq [0,1]$

  hence $1 \in B \Rightarrow 1 - \alpha \in A$

  but $2k\alpha \leq 1 < (2k+1)\alpha \Rightarrow (2k-1)\alpha \leq 1 - \alpha < 2k\alpha$

  contradiction because $A \cap B = \emptyset$, $[(2k-1)\alpha, 2k\alpha) \subseteq B$

- Case 2: $(2k+1)\alpha \leq 1 < (2k+2)\alpha$

  Repeating our previous argument $[2k\alpha, (2k+1)\alpha) \subseteq A$

  and $B = A + \alpha \Rightarrow [(2k+1)\alpha, (2k+2)\alpha) \subseteq B$

  This is a contradiction because there are numbers between 1 and $(2k+2)\alpha$ that belong to B even though $B \subseteq [0,1]$

Comment: We were careful to choose the "maximal" k before proceeding with induction to ensure that we wouldn't run out of bounds (Otherwise the inductive argument would go on forever giving a wrong conclusion.)

Problem 5(Panos)

Πέμπτη, 26 Δεκεμβρίου 2024 12:58 μμ

For $n=3$ (base case) $3^{3+1}=81>64=(3+1)^3$

Inductive hypothesis: $\exists n\geq 3: n^{n+1}>(n+1)^n$

then $n^{n+1}>(n+1)^n \Rightarrow \underline{n+1} > \left(\frac{n+1}{n}\right)^{n+1} \overset{*}{>} \underline{\left(\frac{n+2}{n+1}\right)^{n+1}} \Rightarrow$

$n+1 \geq \left(\frac{n+2}{n+1}\right)^{n+1} \Rightarrow (n+1)^{n+2} \geq (n+2)^{n+1}$ ✓

*: We used the fact

$(n+1)^2 \geq n(n+2) \Rightarrow \frac{n+1}{n} \geq \frac{n+2}{n+1} \Rightarrow \left(\frac{n+1}{n}\right)^{n+1} \geq \left(\frac{n+2}{n+1}\right)^{n+1}$

Therefore by induction: $n^{n+1} \geq (n+1)^n \quad \forall n \geq 3$

Alternatively:

$n^{n+1} > (n+1)^n \Leftrightarrow (n+1)\ln n > n\ln(n+1) \Rightarrow \frac{\ln n}{n} > \frac{\ln(n+1)}{n+1}$

Define $f(x)=\frac{\ln x}{x}$, $x>0$ $\quad f'(x)=\frac{1-\ln x}{x^2}$

so $f'(x)<0 \Leftrightarrow 1<\ln x \Leftrightarrow x>e$

hence f is strictly decreasing on $[e,+\infty)$

Thus $f(n)>f(n+1)$ and we are done.

Problem 8

Πέμπτη, 26 Δεκεμβρίου 2024 1:12 μμ

Base case: $(\sqrt{a_1}-\sqrt{a_2})^2 \geq 0 \Rightarrow \frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2}$ (1)

Inductive hypothesis: $\exists n \geq 2: \frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \quad \forall a_1, a_2, \dots, a_n > 0$

$$\frac{1}{2n}\sum_{i=1}^{2n} a_i = \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n} a_i + \frac{1}{n}\sum_{i=n+1}^{2n} a_i\right) \geq \frac{1}{2}\left[\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=n+1}^{2n} a_i\right)^{\frac{1}{n}}\right] \overset{(1)}{\geq}$$

$$\sqrt{\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\left(\prod_{i=n+1}^{2n} a_i\right)^{\frac{1}{n}}} = \left(\prod_{i=1}^{2n} a_i\right)^{\frac{1}{2n}} \Rightarrow \frac{1}{2n}\sum_{i=1}^{2n} a_i \geq \left(\prod_{i=1}^{2n} a_i\right)^{\frac{1}{2n}} \quad \checkmark$$

Therefore we've proved that if the AM-GM inequality holds for n, it also holds for $2n$.

So we've proved AM-GM by induction for all powers of 2
$(n=2 \longrightarrow n=4 \longrightarrow n=8 \longrightarrow \dots)$

We now need to find a way to go from n to $n-1$.

Inductive hypothesis: $\exists n \geq 2: \frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \quad \forall a_1, a_2, \dots, a_n > 0$

Set $A = \left(\prod_{i=1}^{n-1} a_i\right)^{\frac{1}{n-1}}$

Apply AM-GM for the n terms $a_1, a_2, \dots, a_{n-1}, A$

$$\frac{1}{n}\left(\sum_{i=1}^{n-1} a_i + A\right) \geq \left(\prod_{i=1}^{n-1} a_i \cdot A\right)^{\frac{1}{n}} \Rightarrow \frac{1}{n}\left(\sum_{i=1}^{n-1} a_i + A\right) \geq \left(A^{n-1}\cdot A\right)^{\frac{1}{n}} \Rightarrow$$

$$\sum_{i=1}^{n-1} a_i \geq (n-1)A \Rightarrow \frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq A = \left(\prod_{i=1}^{n-1} a_i\right)^{\frac{1}{n-1}}$$

We have now proved by induction the AM-GM inequality for all $n \in \mathbb{N}$.

Problem 9(Konstantinos)

Πέμπτη, 26 Δεκεμβρίου 2024 1:33 μμ

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

Denote by S the product of the entries inside the 8 squares in the accompanying figure. S will be our invariant.

- Whenever we change signs in a row or column S remains constant because exactly two signs inside the blue rectangles are changed ($S' = (-1)^2 S = S$)
- Whenever we change signs in a parallel to one diagonal, either 2 or 0 signs are changed inside the blue rectangles. Hence again S is constant.

Therefore S is invariant.

At first $S=-1$, but if the grid was full of '1's S would be $S=1$.

It's impossible to get a grid of '1's.

Alternatively, we may consider the sum F of the entries of the 8 squares we used previously.

Each time 2 or 0 signs are changed.

We consider 3 cases:

before	$\boxed{1}\ \boxed{1}$	$\boxed{-1}\ \boxed{1}$	$\boxed{-1}\ \boxed{-1}$
	$\downarrow$	$\downarrow$	$\downarrow$
after	$\boxed{-1}\ \boxed{-1}$	$\boxed{1}\ \boxed{-1}$	$\boxed{1}\ \boxed{1}$
$F'-F$	-4	0	$+4$

We see that $F \pmod 4$ is invariant.

Initially $F \equiv 2 \pmod 4$ so we can't get $F \equiv 0 \pmod 4$.