

Εσωτερικός διαγωνισμός επιλογής για τον IMC 2024
Τμήμα Μαθηματικών Πατρών

24/05/24

Διάρκεια εξέτασης: 3 ώρες.

Problem 1. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+2)^2} < \frac{5}{32}.$$

Problem 2. Let $a, b \in \mathbb{R}$ and $f(x) = a \cos x + b \cos 3x$. If the inequality $f(x) > 1$ has no solutions, prove that $|b| \leq 1$.

Problem 3. Let a_n be a sequence of nonzero real numbers and let $b_n = \frac{a_{n+1}^2}{a_n^2}$. Assume that the series

$$\sum_{n=1}^{\infty} a_n$$

diverges, but the series

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Examine the convergence of the series

$$\sum_{n=1}^{\infty} (e^{b_n} - 1).$$

Problem 4. Suppose that P and Q are $n \times n$ matrices such that $P^2 = P$, $Q^2 = Q$ and $I - (P + Q)$ is invertible. Show that P and Q have the same rank.

Problem 5. Let

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & 0 & 3 \end{pmatrix}.$$

Calculate the trace of the matrix A^n for $n \geq 1$.

Problem 6. The positive divisors of an integer $n > 1$ are $1 = d_1 < d_2 < \dots < d_k = n$. Let $s = d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$. Prove that $s < n^2$ and find all n for which s divides n^2 .

Problem 7. Find all primes p such that $7p + 3^p - 4$ is a square.