SEEMOUS AND IMC PREPARATION, DAY 1, 4/11/2022
ANALYSIS

1. Basic knowledge

Basic theory of real functions, limits, continuity, derivatives.

2. Exercises

Limits and functions

1. Find the limit $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$.

2. Let $a_1,...,a_n$ be positive real numbers. Find the limit

$$\lim_{x\to 0}\left(\frac{a_1^x+...+a_n^x}{n}\right)^{1/x}.$$

3. Does

$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\frac{1}{\cos x}}$$

exist?

4. Let S be the set of rational numbers which are different from $-1,0,1$. Let $f:S\to S$ with $f(x)=x-\frac{1}{x}$. We set $f^{(n)}$ the composition of f with itself n times. Examine whether

$$\bigcap_{n=1}^{\infty} f^{(n)}(S)\neq\emptyset.$$

Continuity

5. Does there exist a continuous function $f:[0,1]\to\mathbb{R}$ that assumes every element of its range an even (finite) number of times?

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. If $\lim_{n\to\infty} f(na)=0$ for every $a>0$, prove that $\lim_{x\to\infty} f(x)=0$.

Derivatives

7. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable infinitely many times. If

$$f\left(\frac{1}{n}\right)=\frac{n^2}{n^2+1},\ n=1,2,3,...$$

calculate $f^{(k)}(0)$ for every $k\geq 1$.

8. (IMC 2019, Day 2, Problem 1) Suppose $f, g : \mathbb{R} \to \mathbb{R}$. Let f be continuous and g differentiable on $\mathbb{R}$. Assume

$$(f(0) - g'(0))(g'(1) - f(1)) > 0.$$

Show that there exists $c \in (0, 1)$ such that $f(c) = g'(c)$.

9. For $x \geq 2$ prove that

$$(x+1)\cos\left(\frac{\pi}{x+1}\right) - x\cos\left(\frac{\pi}{x}\right) > 1.$$

10. Let $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$ be a trigonometric polynomial with $a_i \in \mathbb{R}$. Prove that if $f(x) \leq |\sin x|$ for all $x \in \mathbb{R}$, then

$$\left|\sum_{k=1}^{n} k a_k\right| \leq 1.$$

11. (IMC 2013, Day 1, Problem 2) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f(0) = 0$. Prove that there exists $\xi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

12. (IMC 2012, Day 1, Problem 4) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies $f'(t) > f(f(t))$ for all $t \in \mathbb{R}$. Prove that $f(f(f(t))) \leq 0$ for all $t \geq 0$.

SEEMOUS AND IMC PREPARATION, DAY 2, 02/12/2022
DISCRETE MATHEMATICS

1. Basic knowledge for Discrete mathematics

1.1. **Elementary counting with bijections.** In some problems (of discrete type) we want to prove that two different (finite) sets A and B have the same cardinality (number of their elements). A very nice way to prove that is by constructing a bijection $\phi : A \to B$, i.e. a map which is injection (one-by one) and surjection (onto). If we have such a map, then $|A| = |B|$.

If we find an injection $\phi : A \to B$ then we can only claim that $|A| \leq |B|$. If we find a surjection $\phi : A \to B$ then we can only claim that $|A| \leq |B|$.

1.2. **Additive and multiplicative principles.** There are some important facts from set theory that we need as a background.

1). If $A_1, ..., A_n$ are disjoint sets then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

2. If $\phi : A \to B$ is a map of finite sets and for every $y \in B$ there exists exactly m elements $x \in A$ then $|A| = m|B|$.

In this way we can prove that the number of permutations of a set A with n elements is $n!$. The set of permutations of n elements is the group S_n, thus $|S_n| = n!$.

In the same way we can prove that if $|A| = n$ then A has exactly 2^n different subsets.

1.3. **Inclusion-exclusion principle.** Another useful tool is inclusion–exclusion principle, stating that for any sets $A_1, ..., A_n$ then

$$|A_i \cup A_j| = |A_i| + |A_j| - |A_i \cap A_j|,$$

for any two sets A_i, A_j,

$$|A_i \cup A_j \cup A_k| = |A_i| + |A_j| + |A_k| - |A_i \cap A_j| - |A_i \cap A_k| - |A_j \cap A_k| + |A_i \cap A_j \cap A_k|,$$

for any three sets A_i, A_j, A_k, and more generally

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|.$$

1.4. **Invariants.** Sometimes it's useful to search for invariant quantities. An invariant is a quantity that does't change during a procedure, a game or in a dynamical enviroment. In other words, if there is a repetition, try to search for something that does not change.

1.5. **Box principle (Pigeonhole principle).** If you try to put $n+1$ pigeons in n boxes, then at least one box should contain at least 2 pigeons. And if you try to put $mn+1$ numbers in n sets, then at least one set should contain at least $m+1$ numbers.

2. Exercises

Counting and basic principles

1. How many subsets of $\{1, 2, ..., 10000\}$ contain at least one even number?

2. In a tennis tournament n players participate. Each game is a knock out: the winner continues whereas the loser leaves the tournament. At the end of the last game of the tournament, the last winner wins the tournament and this is the end. How many games took place in total?

3. A partition of n is an ordered sequence $(\lambda_1, ..., \lambda_k)$ such that $\sum_i \lambda_i = n$. We write $p(n)$ for the number of different partitions of n. For instance, $p(4) = 5$ and $p(5) = 7$. Let $o(n)$ count the number of partitions of n with odd parts and $q(n)$ the number of partitions of n with not equal parts. For instance, $o(4) = 2$ and $q(4) = 2$, whereas $o(5) = 3$ and $q(5) = 3$. Prove that for every n we have $o(n) = q(n)$.

4. Let B_n be the number of permutations without fixed points. Then

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - ... + \frac{(-1)^n}{n!}\right).$$

Deduce that

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}.$$

Invariants

5. Let $n \geq 1$ be an odd integer and wite down all the numbers $1, 2, ..., 2n$. Then pick any two numbers a, b, erase them and write down instead $|a - b|$. Prove that an odd number will remain at the end.

6. A circle is divided into six sectors. Then we write the numbers $1, 0, 1, 0, 0, 0$ into the sectors (directed). You may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?

7. Suppose we have four integers a, b, c, d not all four equal. Starting from (a, b, c, d) in each step we replace (a, b, c, d) by $(a - b, b - c, c - d, d - a)$. Then at least one of the quadraple will eventually become arbitrarily large.

8. Let $d(n)$ be the digital sum of $n \geq 1$. Solve $n + d(n) + d(d(n) + d(d(d(n))) + d(d(d(d(n)))) + d(d(d(d(d(n))))) = 2999999999999999999996$.

9. Can we rearrange the integers $1, 1, 2, 2, 3, 3, ..., 1000002, 1000002$ such that there are exactly $n - 1$ numbers between any n's?

Pigeonhole

10. Let $S_n = \{1, 2, 3, ..., 2n\}$. If we pick $n + 1$ numbers from S_n, prove that one of them is divisible by another.

11. (IMC 2008, Day 2, Problem 4) Let $f(x), g(x)$ be two nonconstant polynomials in $\mathbb{Z}[x]$ such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if $f(x) - 2008$ has at least 81 distinct integer roots, then the degree of $g(x)$ is at least 5.

12. Given n distinct real numbers $a_1, ..., a_n$ and $M > 0, T > 1$ there is an s such that $M \leq s \leq MT^n$ and

$$|e^{ia_j s} - 1| < \frac{1}{T}$$

for all $j = 1, ..., n$.

SEEMOUS AND IMC PREPARATION, DAY 3, 13/1/2023
ANALYSIS

1. Basic knowledge

Basic theory of integrals (Calculus 1 and 2).

2. Exercises

Functional equations

1. Find all continuous functions $f : [0, 1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)dx = \frac{1}{3} + \int_0^1 f(x^2)^2 dx.$$

2. (IMC 2022, P1, Day 1) Let $f : [0, 1] \to (0, \infty)$ be an integrable function such that $f(x)f(1-x) = 1$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f(x)dx \geq 1.$$

3. (SEEMOUS 2013, P1) Find all continuous functions $f : [1, 8] \to \mathbb{R}$ such that

$$\int_1^2 f^2(x^3)dx + 2\int_1^2 f(x^3)dx = \frac{2}{3}\int_1^8 f(x)dx - \int_1^2 (x^2-1)^2 dx.$$

4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^\pi x f(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx.$$

5. (SEEMOUS 2013, P3) Find the maximum value of

$$\int_0^1 |f'(x)|^2 |f(x)| \frac{1}{\sqrt{x}} dx$$

over all continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ and

$$\int_0^1 |f'(x)|^2 dx \leq 1.$$

6. Let f be a continuously differentiable function $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ and $0 < f'(x) \leq 1$ for all $x \in [0, 1]$. Prove that

$$\left(\int_0^1 f(x)dx\right)^2 \geq \int_0^1 f(x)^3 dx.$$

Give an example where equality holds.

7. Let $f : [1, \infty) \to [1, \infty)$ be a continuous function and let $c > 0$ be a constant such that

$$\int_1^t f(x)dx \leq ct^2$$

for all $t > 1$. Prove that

$$\int_1^\infty \frac{1}{f(x)} dx = \infty.$$

Computations

8. Calculate the integral

$$\int_0^{\frac{\pi}{2}} \log(\sin(x))dx.$$

9. For every $a \in \mathbb{R}$ prove that

$$\int_0^\pi \log\left((\sin a \cos x)^2 + (\cos a \sin x)^2\right) dx \leq -\pi \log 2.$$

10. Compute the integral

$$\int_0^\pi \frac{x \sin x}{1 + \sin^2 x} dx.$$

11. Let $p(x)$ be a polynomial with real coefficients. Calculate the integral

$$\int_0^\infty e^{-x} p(x) dx.$$

SEEMOUS and IMC preparation seminar, Day 4

George Soukaras

March 17, 2023

1 Theory/Background

Everything from Calculus I, Calculus II, Linear Algebra I, Linear Algebra II.

2 Problems of SEEMOUS 2023

Today we will discuss the following problems from the SEEMOUS 2023 Competition.

Problem 1 Prove that if A and B are $n \times n$ square matrices with complex entries satisfying

$$A = AB - BA + A^2B - 2ABA + BA^2 + A^2BA - ABA^2$$

then $\det(A) = 0$.

Problem 2 For the sequence

$$S_n = \frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}}$$

find

$$\lim_{n\to\infty} n\left(n(\ln(1+\sqrt{2}) - S_n) - \frac{1}{2\sqrt{2}(\sqrt{2}+1)}\right)$$

Problem 3 Prove that if A is $n \times n$ square matrix with complex entries such that $A + A^* = A^2A^*$, then $A = A^*$.(For any matrix M, denote by $M^* = \overline{M}^t$ the conjugate transpose of M.)

Problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly decreasing function such that $f([0,1]) \subseteq [0,1]$.
(i) For all $n \in \mathbb{N} \setminus \{0\}$, prove that there exists $a_n \in (0,1)$, solution of the equation

$$f(x) = x^n$$

Moreover, if (a_n) is the sequence defined as above, prove that $\lim_{n\to\infty} a_n = 1$.

(ii) Suppose f has a continuous derivative, with $f(1) = 0$ and $f'(1) < 0$. For any $x \in \mathbb{R}$, we define

$$F(x) = \int_x^1 f(t)\,dt$$

Study the convergence of the series $\sum_{n=1}^{\infty} F(a_n)^{\alpha}$, with $\alpha \in \mathbb{R}$.

3 Problems from previous SEEMOUS competitions

Problem 5 (SEEMOUS 2020, P2) Let $k > 1$ be a real number. Calculate:

$$(a) L = \lim_{n\to\infty} \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n dx$$

$$(b) \lim_{n\to\infty} n \left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x} + k - 1} \right)^n dx \right]$$

Problem 6 (SEEMOUS 2020, P4) Consider $0 < \alpha < T, D = \mathbb{R} \setminus \{kT + \alpha | k \in \mathbb{Z}\}$, and let $f : D \to \mathbb{R}$ a T-periodic and differentiable function which satisfies $f' > 1$ on $(0, \alpha)$ and

$$f(0) = 0, \lim_{x\to\alpha^-} f(x) = +\infty, \lim_{x\to\alpha^-} \frac{f'(x)}{f^2(x)} = 1$$

(i) Prove that for every $n \in \mathbb{N} \setminus 0$, the equation $f(x) = x$ has a unique solution in the interval $(nT, nT + \alpha)$, denoted x_n.

(ii) let $y_n = nT + a - x_n$ and $z_n = \int_0^{y_n} f(x)\, dx$. Prove that $\lim_{n\to\infty} y_n = 0$ and study the convergence of the series $\sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty} z_n$.

IMC preparation seminar, Day 5

April 6, 2023

1 Theory/Background

Divisibility, prime numbers, congruences and Euler's theorem, Wilson's theorem, Structure of $\mathbb{Z}_n$, sequences, multiplicative functions.

2 Problems

Problem 1 Let k be an even number. Is it possible to write 1 as the sum of the reciprocals of k odd integers?

Problem 2 Player A has chosen five numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$. If he told Claudia what the product of the chosen numbers was, that would not be enough information for Player B to figure out whether the sum of the chosen numbers was even or odd. What is the product of the chosen numbers?

Problem 3 Let a and b be distinct positive integers such that $ab(a + b)$ is divisible by $a^2 + ab + b^2$. Prove that $|a - b| > \sqrt[3]{ab}$.

Problem 4 Find all primes p and q such that $p + q = (p - q)^3$.

Problem 5 Find all $n \geq 1$ such that

$$n! \mid \prod_{p<q\leq n} (p + q).$$

Problem 6 When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.

Problem 7 Suppose that x is a real number for which

$$\left\lfloor x + \frac{19}{100} \right\rfloor + \left\lfloor x + \frac{20}{100} \right\rfloor ... + \left\lfloor x + \frac{91}{100} \right\rfloor = 546.$$

Find $\lfloor 100x \rfloor$.

Problem 8 Find all positive integers n for which $n! + 5$ is a perfect cube.

Problem 9 (IMC 2020, P6) Find all prime numbers p for which there exists a unique $a \in \{1, 2, ..., p\}$ such that $a^3 - 3a + 1$ is divisible by p.

Problem 10 (IMC 2013, P5) Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that

$$\sum_{n=1}^{\infty} a_n^p$$

converges if and only if p is not a prime?

Problem 11 Find divisibility rules for 7 and for 17.

Problem 12 (IMC 2022, P6) Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, ..., x_{p-1})$ of the numbers $(1, 2, ..., p-1)$ such that

$$x_1x_2 + x_2x_3 + ... + x_{p-2}x_{p-1} \equiv 2 \mod p.$$

Problem 13 Find all non-negative integers x, y, z satisfying $2^x + 3^y = z^2$.

Problem 14 Find all non-negative integers x, y satisfying $x^2 + 17y^2 = 3$.

Problem 15 Find all primes p and positive integers x, y satisfying

$$\frac{xy^3}{x+y} = p.$$

Problem 16 (IMO shortlist 1986) The set $S = \{2, 5, 13\}$ has the property that for all distinct $x, y \in S$

$$xy - 1 = \Box.$$

Show that for all $n \notin S$ the set $S \cup \{n\}$ does not have this property.

IMC PREPARATION-LINEAR ALGEBRA

5th of May 2023

Basic theory

- A matrix A is called square if its dimension is n x n for $n \in \mathbb{N}$. The set of all m x n with elements in a ring R will be denoted by $M_{m,n}(R)$ (for us $R = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$). The element of A that is in the i-th row and j-th column will be denoted by a_{ij} and we write $A = (a_{ij})$. A^t is the transpose matrix of A i.e. $A^t = (a_{ji})$. A^* is the conjugate transpose of A i.e. $A^* = (\overline{a_{ji}})$. The identity matrix (of any size) will be denoted by I. A matrix $A \in M_{n,n}(R)$, is invertible if there exists a matrix $B \in M_{n,n}(R)$ such that $AB = BA = I$. B is unique, called the inverse matrix of A and denoted by A^{-1}. If all elements above the main diagonal or below the main diagonal of a matrix A are zero, we call A lower and upper triangular respectively. A matrix that is both upper and lower triangular is called diagonal.
- Given a square matrix A, an eigenvector $v \in M_{n,1}(\mathrm{F})$ and its corresponding eigenvalue $\lambda \in \mathrm{F} \setminus \{0\}$ satisfy the equation $Av = \lambda v$.
- The trace of a square matrix A is the sum of its diagonal entries, i.e., $\operatorname{tr}(A) = \sum_i a_{ii}$. The determinant of a square matrix A will be denoted by $\det(A)$.
- A square matrix A is called idempotent if $A^2 = A$. It is called nilpotent if there exists an integer m such that $A^m = 0$.
- The rank of a matrix A is the dimension of the vector space spanned by its columns (or equivalently, its rows) and will be denoted by $\operatorname{rank}(A)$

  Theorem (Sylvester rank inequality)**.** *For matrices A, B where A has n columns and B has n rows:*

  $$\operatorname{rank}(AB) \geq \operatorname{rank}(A) + \operatorname{rank}(B) - n$$

  Theorem (Frobenius inequality)**.** *For matrices of appropriate size:*

  $$\operatorname{rank}(ABC) \geq \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B)$$

- The characteristic polynomial of a matrix A is the polynomial $\chi_A(x) = \det(xI - A)$, where I is the identity matrix. Its roots are the eigenvalues of A. The minimal polynomial of A is the monic polynomial of lowest degree that annihilates A and will be denoted by $m_A(x)$.

  Theorem (Cayley-Hamilton)**.**

  $$\chi_A(A) = 0$$

- A square matrix A is symmetric if $A = A^t$, Hermitian if $A = A^*$, orthogonal if $A^tA = AA^t = I$, unitary if $A^*A = AA^* = I$, and normal if $AA^* = A^*A$. If there exists an invertible P and diagonal D such that $A = PDP^{-1}$, we call A diagonalizable. A is diagonalizable $\iff$ its minimal polynomial is a product of linear factors. A matrix A is normal $\iff$ it is unitarily diagonalizable.

Problems

1. For any integer $n \geq 2$ and $A, B \in M_{n,n}(\mathbb{R})$ that satisfy the equation $(A + B)^{-1} = A^{-1} + B^{-1}$, show that $\det(A) = \det(B)$. Does the same conclusion follow for matrices with complex entries?
2. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

3. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric $M \in M_{2,2}(\mathbb{R})$ satisfying $\text{tr}(M) = a$ and $\det(M) = b$.

4. For an idempotent matrix A, show that $\text{rank}(A) = \text{tr}(A)$.

5. For $A \in M_{2,2}(\mathbb{Z})$ that satisfies

$$\det(A^3 + A^2 + A + I) = 1$$

show that $\det(A + I) = \det(A^2 + I) = 1$. What are the possible values of $\det(A)$ and $\text{tr}(A)$?

6. Find all $A \in M_{n,n}(\mathbb{C})$ such that $A^{2023} = A^*A - AA^*$.

7. Let $A, B \in M_{n,n}(\mathbb{C})$ such that $A^*B = O$. Show that $\text{rank}(A^*A + B^*B) \leq \text{rank}(AA^* + BB^*)$.

8. Let $A, B \in M_{n,n}(\mathbb{R})$ such that $A \neq B$, $A^3 = B^3$ and $A^2B = B^2A$. Can $A^2 + B^2$ be invertible?

9. Calculate the determinant of the n x n matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}$$

10. Let $A, B \in M_{2,2}(\mathbb{Z})$ such that $A, A+B, A+2B, A+3B, A+4B$ are invertible matrices such that their inverses also have integer entries. Show that $A+5B$ is also invertible and its inverse has integer entries.

11. For $n \in \mathbb{N}$, let d_n be the greatest common divisor of the elements of the matrix $A^n - I$, where

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

Show that $\lim_{n \to +\infty} d_n = +\infty$.

12. Find all matrices $A \in M_{n,n}(\mathbb{R})$ whose eigenvalues are all real and which satisfy the relation $A + A^k = A^t$ for some $k \geq n$.

13. Let $A_1, A_2, ..., A_k \in M_{n,n}(\mathbb{C})$ be idempotent matrices such that $A_iA_j = -A_jA_i$ for all $i \neq j$. Show that at least one of the given matrices has rank $\leq \frac{n}{k}$.

14. Determine whether there exists an odd positive integer n, matrices $A, B \in M_{n,n}(\mathbb{Z})$ such that:

   (a) $\det(B) = 1$
   (b) $AB = BA$
   (c) $A^4 + 4A^2B^2 + 16B^4 = 2019I$