• ΤΕΛΕΣΤΗΣ $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}.$

$$\boxed{\frac{\partial}{\partial x} \rightarrow \frac{d}{dx}}$$

• ΤΕΛΕΣΤΗΣ LAPLACE : $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \Rightarrow$$

$$\Rightarrow \boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}} \Leftarrow$$

ΕΦΑΡΜΟΓΗ

Να βρεθεί το $\nabla^2 f(x,y) = ?$ όπου.

$f(x,y) = x^2 + xy + y^2.$ → Βαθμωτή συνάρτηση.

$$\nabla^2 f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} + \frac{\partial^2 f(x,y)}{\partial z^2} =$$

$$= \underbrace{\frac{\partial^2}{\partial x^2}(x^2+xy+y^2)}_{I_1} + \underbrace{\frac{\partial^2}{\partial y^2}(x^2+xy+y^2)}_{I_2} + \underbrace{\frac{\partial^2}{\partial z^2}(x^2+xy+y^2)}_{I_3} \Rightarrow$$

$$\Rightarrow \nabla^2 f(x,y) = I_1 + I_2 + I_3 = 4$$

ΠΑΡΑΤΗΡΗΣΕΙΣ

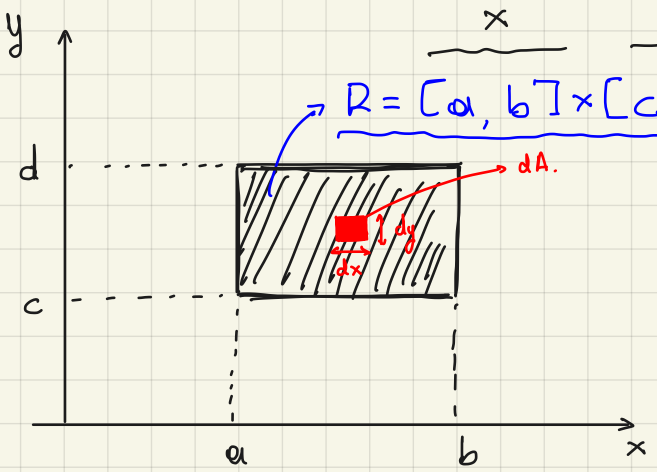
$$I_1 = \frac{\partial^2}{\partial x^2}(x^2+xy+y^2) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x^2+xy+y^2)\right) = \frac{\partial}{\partial x}(2x+y) = 2$$

$$I_2 = \frac{\partial^2}{\partial y^2}(x^2+xy+y^2) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(x^2+xy+y^2)\right) = \frac{\partial}{\partial y}(x+2y) = 2.$$

$$I_3 = \frac{\partial^2}{\partial z^2}(x^2+xy+y^2) = 0$$

ΔΙΠΛΑ ΟΛΟΚΛΗΡΩΜΑΤΑ

- Διπλό ολοκλήρωμα σε ορθογώνιο:



$$R = \underbrace{[a, b]}_{x} \times \underbrace{[c, d]}_{y}.$$

Ορίζω το διπλό ολοκλήρωμα

$$\boxed{\int_R f(x,y)\, \boxed{dA}}$$

πάνω σε ένα ορθογώνιο $R = [a,b] \times [c,d]$

dA : στοιχειώδες κομμάτι εμβαδού του ορθογωνίου R.

$dA = dx\,dy$

$$\int_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_R f(x,y)\, dA.$$

ΕΦΑΡΜΟΓΗ

Να βρεθεί το ολοκλήρωμα $\int_R 4x^2y^2\, dA$ όπου $R = \underbrace{[-1, 1]}_{x} \times \underbrace{[-2, 2]}_{y}$.

$$\int_{-2}^{2} \boxed{\int_{-1}^{1} 4x^2y^2\, dx}\, dy = \int_{-2}^{2} \frac{8}{3} y^2\, dy = \frac{8}{3} \left. \frac{y^3}{3} \right|_{-2}^{2} = (*)$$

$$\int ax\, dx = a \int x\, dx$$

$$\int_{-1}^{1} 4x^2y^2\, dx = 4y^2 \int_{-1}^{1} x^2\, dx = 4y^2 \left. \frac{x^3}{3} \right|_{-1}^{1} =$$

$$= \frac{4}{3} y^2 \left(1^3 - (-1)^3\right) = \frac{8}{3} y^2.$$

$\underset{=}{(*)}\ \frac{8}{9}\left(2^3-(-2)^3\right)=\frac{8}{9}(8+8)=\frac{8\cdot 16}{9}=\boxed{\frac{128}{9}}$

B' ΤΡΟΠΟΣ

$$\int_{-1}^{1}\boxed{\int_{-2}^{2}4x^2y^2\,dy}\,dx=\int_{-1}^{1}\frac{64}{3}x^2\,dx=\frac{64}{3}\int_{-1}^{1}x^2\,dx=\frac{64}{3}\frac{x^3}{3}\Big|_{-1}^{1}\underset{=}{(*)}$$

$$4x^2\int_{-2}^{2}y^2\,dy=4x^2\frac{y^3}{3}\Big|_{-2}^{2}=\frac{4}{3}x^2\left(2^3-(-2)^3\right)=\frac{4\cdot 16}{3}x^2=\boxed{\frac{64}{3}x^2}$$

$\underset{=}{(*)}\ \frac{64}{9}\left(1^3-(-1)^3\right)=\frac{64}{9}\cdot(2)=\boxed{\frac{128}{9}}$

ΕΦΑΡΜΟΓΗ

$$\int_{1}^{2}\int_{3}^{4}x^2+2xy\;dx\,dy=\int_{1}^{2}\left[\int_{3}^{4}x^2\,dx+\int_{3}^{4}2xy\,dx\right]dy=$$

$$=\int_{1}^{2}\frac{x^3}{3}\Big|_{3}^{4}+\cancel{2}y\frac{x^2}{\cancel{2}}\Big|_{3}^{4}\;dy=\int_{1}^{2}\frac{1}{3}\left(4^3-3^3\right)+y\cdot\left(4^2-3^2\right)\,dy=$$

$$=\int_{1}^{2}\frac{64-27}{3}+7y\;dy=\int_{1}^{2}\frac{37}{3}+7y\;dy=$$

$$=\int_{1}^{2}\frac{37}{3}\,dy+\int_{1}^{2}7y\,dy=\frac{37}{3}y\Big|_{1}^{2}+7\frac{y^2}{2}\Big|_{1}^{2}=$$

$$=\frac{37}{3}(2-1)+\frac{7}{2}\left(2^2-1^2\right)=\frac{37}{3}+\frac{21}{2}=$$

$$=\frac{74+63}{6}=\boxed{\frac{137}{6}}\Leftarrow$$

$$\int_3^4\int_1^2 x^2+2xy\;dy\,dx = \int_3^4\left[\int_1^2 x^2\,dy + \int_1^2 2xy\,dy\right]dx =$$

$$= \int_3^4\left(x^2y\Big|_1^2 + \cancel{2}x\,\frac{y^2}{\cancel{2}}\Big|_1^2\right)dx = \int_3^4\left[x^2(2-1) + x\,(2^2-1^2)\right]dx =$$

$$= \int_3^4 x^2+3x\;dx = \int_3^4 x^2\,dx + \int_3^4 3x\,dx = \frac{x^3}{3}\Big|_3^4 + 3\,\frac{x^2}{2}\Big|_3^4 =$$

$$= \frac{1}{3}\left(4^3-3^3\right) + \frac{3}{2}\left(4^2-3^2\right) = \ldots\ldots = \boxed{\frac{137}{6}}$$

ΕΦΑΡΜΟΓΗ

Υπολογίστε τον όγκο που βρίσκεται κάτω από το γράφημα της $f(x,y) = x^2+4y^2$ και πάνω από το ορθογώνιο $R = \underbrace{[-1,1]}_{x}\times\underbrace{[0,2]}_{y}$.

$$V = \int_{-1}^1\int_0^2 x^2+4y^2\;dy\,dx = \int_{-1}^1\left[\int_0^2 x^2\,dy + \int_0^2 4y^2\,dy\right]dx =$$

$$= \int_{-1}^1\left(x^2y\Big|_0^2 + 4\,\frac{y^3}{3}\Big|_0^2\right)dx = \int_{-1}^1\left[x^2(2-0) + \frac{4}{3}\left(2^3-0^3\right)\right]dx =$$

$$= \int_{-1}^1\left(2x^2 + \frac{32}{3}\right)dx = 2\cdot\frac{x^3}{3}\Big|_{-1}^1 + \frac{32}{3}\,x\Big|_{-1}^1 =$$

$$= \frac{2}{3}\left(1^3-(-1)^3\right) + \frac{32}{3}\left(1-(-1)\right) = \frac{4}{3} + \frac{64}{3} = \boxed{\frac{68}{3}}$$

ΕΦΑΡΜΟΓΗ

Να υπολογίσετε το διπλό ολοκλήρωμα:

$$I = \iint_R \sin(x+y)\,dx\,dy$$ και όπου R το ορθογώνιο

$\underbrace{[0,1]}_{x} \times \underbrace{[0, \pi/2]}_{y}$.

$$\int_0^{\pi/2}\int_0^1 \sin(x+y)\,dx\,dy = \int_0^1\int_0^{\pi/2} \sin(x+y)\cdot dy\,dx.$$

$$\int_0^1 -\cos(x+y)\Big|_0^{\pi/2} dx. =$$

$$-\int_0^1 \left(\cos\left(x+\pi/2\right) - \cos x\right) dx =$$

$$= -\left[\int_0^1 \cos\left(x+\pi/2\right) dx - \int_0^1 \cos x\,dx\right] =$$

$$= -\left[\sin\left(x+\pi/2\right)\Big|_0^1 - \sin x\Big|_0^1\right] =$$

$$= -\left[\left(\sin\left(1+\pi/2\right) - \cancel{\sin \pi/2}^{\,1}\right) - \left(\sin 1 - \cancel{\sin(0)}^{\,0}\right)\right] =$$

$$= -\sin\left(1+\pi/2\right) + 1 + \sin 1 = \boxed{1+\sin 1 - \sin\left(1+\pi/2\right)}$$

$$\int_0^{\pi/2} \sin(x+y)\,dy =$$

$$= -\cos(x+y)\Big|_0^{\pi/2}$$

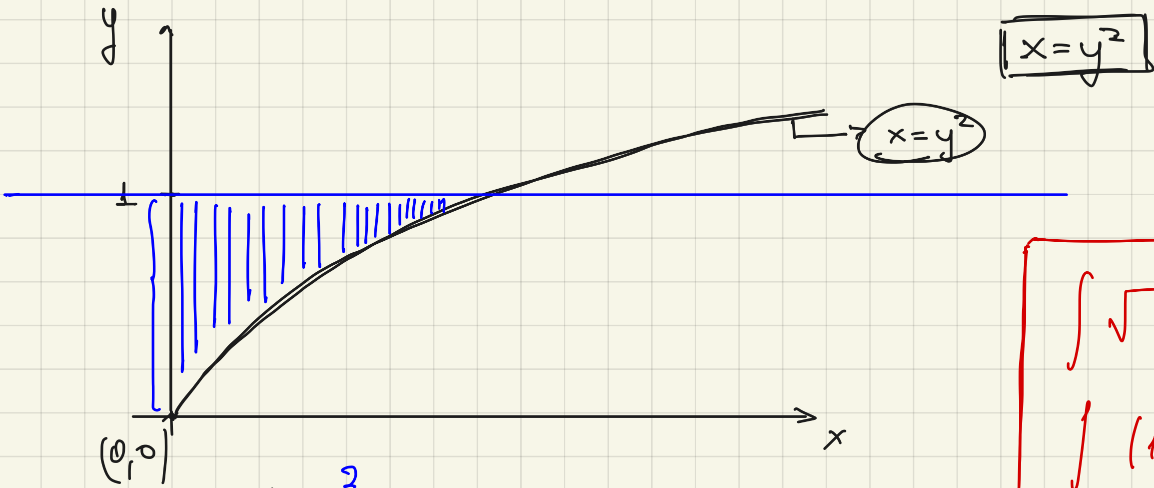
$$\frac{\partial\left(-\cos(x+y)\right)}{\partial y} = -\left(-\sin(x+y)\right)\frac{\partial(x+y)}{\partial y}$$

$$= \boxed{\sin(x+y)}$$

ΕΦΑΡΜΟΓΗ

Να υπολογιστεί το ολοκλήρωμα $I = \iint_D \sqrt{1+y^3}\, dx\, dy \Leftarrow$

όπου $D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 ,\ 0 \le x \le y^2 \} \Leftarrow$



$$I = \int_0^1 \int_0^{y^2} \sqrt{1+y^3}\, dx\, dy =$$

$$\int \sqrt{1+y^3}\, dx = \int (1+y^3)^{1/2} dx = (1+y^3)^{1/2} \int dx = (1+y^3)^{1/2} \cdot x$$

$$= \int_0^1 \sqrt{1+y^3}\, x \Big|_0^{y^2} dy = \int_0^1 \sqrt{1+y^3}\,(y^2-0)\, dy$$

$$= \int_0^1 y^2 \cdot \sqrt{1+y^3}\, dy$$

$$\int y^2 \sqrt{1+y^3}\, dy =$$

Θέτω $\boxed{1+y^3 = u} \Rightarrow$

$\Rightarrow 3y^2 dy = du \Rightarrow$

$\Rightarrow \boxed{y^2 dy = \frac{du}{3}}$

$$\int \sqrt{u}\, \frac{du}{3} = \frac{1}{3} \int \sqrt{u}\, du$$

$$= \frac{1}{3} \int u^{1/2} du = \frac{1}{3}\frac{2}{3} u^{3/2} = \frac{2}{9}(1+y^3)^{3/2}$$

$$= \frac{2}{9}(1+y^3)^{3/2} \Big|_0^1 =$$

$$= \frac{2}{9}\left[(1+1^3)^{3/2} - (1+0^3)^{3/2} \right]$$

$$= \frac{2}{9}\left(2^{3/2} - 1 \right)$$