The 4×4 homogeneous coordinate transformation matrix, ${}^{i-1}\mathbf{A}_i$, which describes the spatial relationship between the ith and the $(i-1)$th link coordinate frames. It relates a point fixed in link i expressed in homogeneous coordinates with respect to the ith coordinate system to the $(i-1)$th coordinate system.

The Lagrange-Euler equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = \tau_i \qquad i = 1, 2, \ldots, n \tag{3.2-1}$$

where

L = lagrangian function = kinetic energy K − potential energy P

K = total kinetic energy of the robot arm

P = total potential energy of the robot arm

q_i = generalized coordinates of the robot arm

$\dot{q}_i$ = first time derivative of the generalized coordinate, q_i

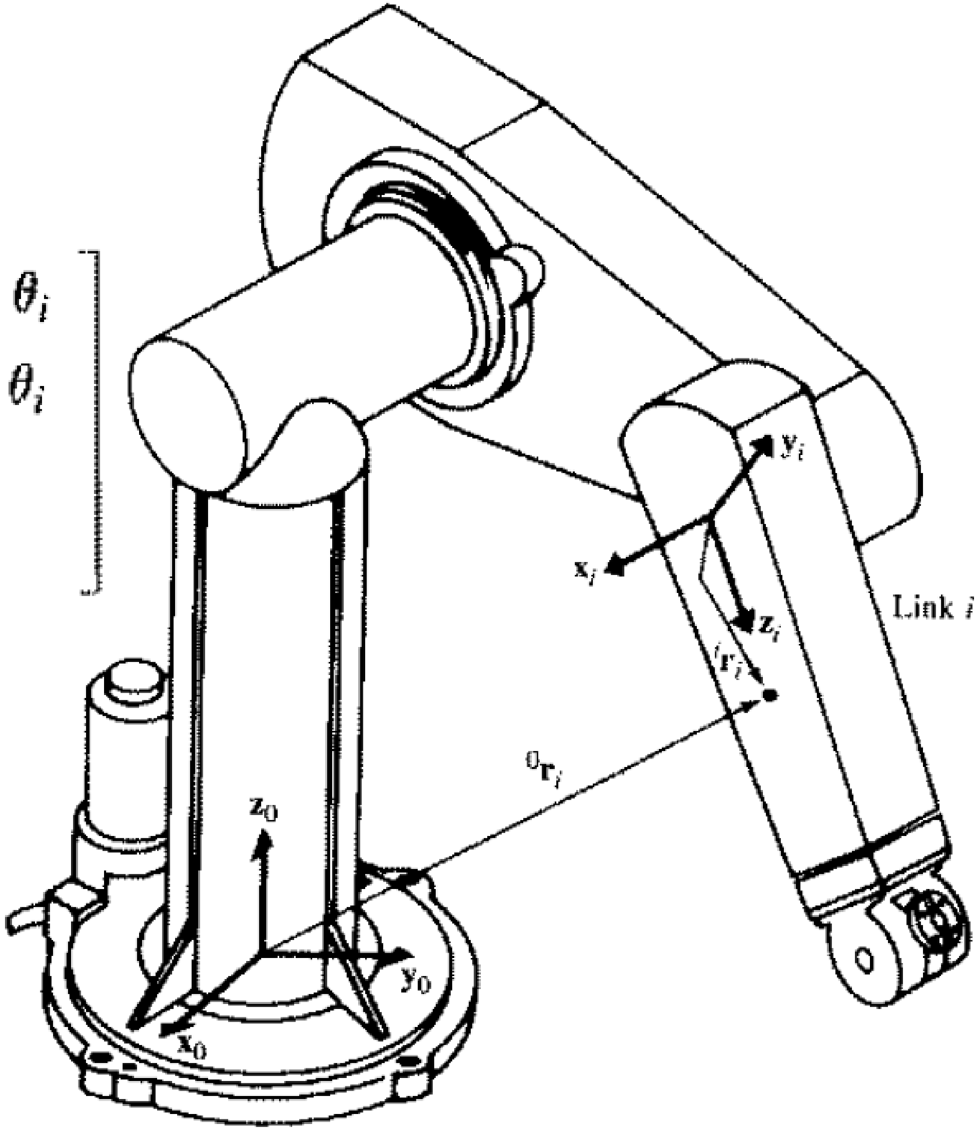
τ_i = generalized force (or torque) applied to the system at joint i to drive link i

$$ {}^{i}\mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} $$

$$ {}^{0}\mathbf{r}_i = {}^{0}\mathbf{A}_i \, {}^{i}\mathbf{r}_i $$

$$ {}^{0}\mathbf{A}_i = {}^{0}\mathbf{A}_1 \, {}^{1}\mathbf{A}_2 \cdots {}^{i-1}\mathbf{A}_i $$

$$ {}^{i-1}\mathbf{A}_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



$$
\begin{aligned}
{}^0\mathbf{v}_i \equiv \mathbf{v}_i &= \frac{d}{dt}({}^0\mathbf{r}_i) = \frac{d}{dt}({}^0\mathbf{A}_i\ {}^i\mathbf{r}_i) \\
&= {}^0\dot{\mathbf{A}}_1\ {}^1\mathbf{A}_2 \cdots {}^{i-1}\mathbf{A}_i\ {}^i\mathbf{r}_i + {}^0\mathbf{A}_1\ {}^1\dot{\mathbf{A}}_2 \cdots {}^{i-1}\mathbf{A}_i\ {}^i\mathbf{r}_i + \cdots \\
&\quad + {}^0\mathbf{A}_1 \cdots {}^{i-1}\dot{\mathbf{A}}_i\ {}^i\mathbf{r}_i + {}^0\mathbf{A}_i\ {}^i\dot{\mathbf{r}}_i = \left(\sum_{j=1}^{i} \frac{\partial\, {}^0\mathbf{A}_i}{\partial q_j} \dot{q}_j \right) {}^i\mathbf{r}_i
\end{aligned}
$$

$$
\mathbf{Q}_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q}_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial\, {}^{i-1}\mathbf{A}_i}{\partial q_i} = \mathbf{Q}_i\ {}^{i-1}\mathbf{A}_i
$$

$$\frac{\partial\, {}^{i-1}\mathbf{A}_i}{\partial \theta_i} = \begin{bmatrix} -\sin\theta_i & -\cos\alpha_i \cos\theta_i & \sin\alpha_i \cos\theta_i & -a_i \sin\theta_i \\ \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\equiv \mathbf{Q}_i\, {}^{i-1}\mathbf{A}_i$$

$$\frac{\partial\, {}^{0}\mathbf{A}_i}{\partial q_j} = \begin{cases} {}^{0}\mathbf{A}_1\, {}^{1}\mathbf{A}_2 \cdots {}^{j-2}\mathbf{A}_{j-1}\mathbf{Q}_j\, {}^{j-1}\mathbf{A}_j \cdots {}^{i-1}\mathbf{A}_i & \text{for } j \leqslant i \\ 0 & \text{for } j > i \end{cases}$$

$$\mathbf{U}_{ij} = \begin{cases} {}^{0}\mathbf{A}_{j-1}\,\mathbf{Q}_j\, {}^{j-1}\mathbf{A}_i & \text{for } j \leqslant i \\ 0 & \text{for } j > i \end{cases}$$

$$\mathbf{v}_i = \left(\sum_{j=1}^{i} \mathbf{U}_{ij}\dot{q}_j \right) {}^{i}\mathbf{r}_i$$

$$\frac{\partial \mathbf{U}_{ij}}{\partial q_k} \triangleq \mathbf{U}_{ijk} = \begin{cases} {}^0\mathbf{A}_{j-1}\mathbf{Q}_j\,{}^{j-1}\mathbf{A}_{k-1}\mathbf{Q}_k\,{}^{k-1}\mathbf{A}_i & i \geqslant k \geqslant j \\ {}^0\mathbf{A}_{k-1}\mathbf{Q}_k\,{}^{k-1}\mathbf{A}_{j-1}\mathbf{Q}_j\,{}^{j-1}\mathbf{A}_i & i \geqslant j \geqslant k \\ 0 & i < j \text{ or } i < k \end{cases}$$

$$\frac{\partial \mathbf{U}_{11}}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}(\mathbf{Q}_1\,{}^0\mathbf{A}_1) = \mathbf{Q}_1\mathbf{Q}_1\,{}^0\mathbf{A}_1$$

Kinetic Energy of a Robot Manipulator

$$dK_i = \tfrac{1}{2}(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\,dm$$

$$= \tfrac{1}{2}\ \text{trace}\ (\mathbf{v}_i\mathbf{v}_i^T)\,dm = \tfrac{1}{2}\ \text{Tr}\ (\mathbf{v}_i\mathbf{v}_i^T)\,dm$$

$$dK_i = \tfrac{1}{2}\,\mathrm{Tr}\left[\sum_{p=1}^{i} \mathbf{U}_{ip}\dot{q}_p\, {}^i\mathbf{r}_i \left(\sum_{r=1}^{i} \mathbf{U}_{ir}\dot{q}_r\, {}^i\mathbf{r}_i\right)^T\right] dm$$

$$= \tfrac{1}{2}\,\mathrm{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} \mathbf{U}_{ip}\, {}^i\mathbf{r}_i\, {}^i\mathbf{r}_i^T \mathbf{U}_{ir}^T \dot{q}_p \dot{q}_r\right] dm$$

$$= \tfrac{1}{2}\,\mathrm{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} \mathbf{U}_{ip}(\, {}^i\mathbf{r}_i\, dm\, {}^i\mathbf{r}_i^T)\, \mathbf{U}_{ir}^T \dot{q}_p \dot{q}_r\right]$$

$$K_i = \int dK_i = \tfrac{1}{2}\mathrm{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} \mathbf{U}_{ip}\left(\int {}^i\mathbf{r}_i\, {}^i\mathbf{r}_i^T\, dm\right) \mathbf{U}_{ir}^T \dot{q}_p \dot{q}_r\right]$$

$$\mathbf{J}_i = \int {}^i\mathbf{r}_i\, {}^i\mathbf{r}_i^T dm = \begin{bmatrix} \int x_i^2\, dm & \int x_i y_i\, dm & \int x_i z_i\, dm & \int x_i\, dm \\ \int x_i y_i\, dm & \int y_i^2\, dm & \int y_i z_i\, dm & \int y_i\, dm \\ \int x_i z_i\, dm & \int y_i z_i\, dm & \int z_i^2\, dm & \int z_i\, dm \\ \int x_i\, dm & \int y_i\, dm & \int z_i\, dm & \int dm \end{bmatrix}$$

$$K = \sum_{i=1}^{n} K_i = ½\sum_{i=1}^{n} \mathrm{Tr}\left(\sum_{p=1}^{i}\sum_{r=1}^{i} \mathbf{U}_{ip}\mathbf{J}_i\mathbf{U}_{ir}^T\dot{q}_p\,\dot{q}_r\right)$$

$$= ½\sum_{i=1}^{n}\sum_{p=1}^{i}\sum_{r=1}^{i}[\mathrm{Tr}\,(\mathbf{U}_{ip}\,\mathbf{J}_i\,\mathbf{U}_{ir}^T)\dot{q}_p\,\dot{q}_r]$$

Potential Energy of a Robot Manipulator

$$P_i = -\,m_i\mathbf{g}\,{}^0\bar{\mathbf{r}}_i = -m_i\mathbf{g}({}^0\mathbf{A}_i\,{}^i\bar{\mathbf{r}}_i) \qquad i = 1, 2, \ldots, n$$

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} -\,m_i\mathbf{g}\,({}^0\mathbf{A}_i\,{}^i\bar{\mathbf{r}}_i)$$

$$\mathbf{g} = (g_x,\ g_y,\ g_z,\ 0)$$

Motion Equations of a Manipulator

$$L = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{i} [\mathrm{Tr}\,(\mathbf{U}_{ij}\,\mathbf{J}_i\,\mathbf{U}_{ik}^T)\dot{q}_j\,\dot{q}_k] + \sum_{i=1}^{n} m_i \mathbf{g}({}^0\mathbf{A}_i\,{}^i\bar{\mathbf{r}}_i)$$

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$= \sum_{j=i}^{n} \sum_{k=1}^{j} \mathrm{Tr}\,(\mathbf{U}_{jk}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T)\,\ddot{q}_k + \sum_{j=i}^{n} \sum_{k=1}^{j} \sum_{m=1}^{j} \mathrm{Tr}\,(\mathbf{U}_{jkm}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T)\,\dot{q}_k\dot{q}_m - \sum_{j=i}^{n} m_j \mathbf{g}\mathbf{U}_{ji}\,{}^j\bar{\mathbf{r}}_j$$

$$\tau_i = \sum_{k=1}^{n} D_{ik}\ddot{q}_k + \sum_{k=1}^{n} \sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m + c_i \qquad i = 1, 2, \ldots, n$$

$$\boldsymbol{\tau}(t) = \mathbf{D}(\mathbf{q}(t))\,\ddot{\mathbf{q}}(t) + \mathbf{h}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{c}(\mathbf{q}(t))$$

$$D_{ik} = \sum_{j=\max(i,k)}^{n} \mathrm{Tr}\,(\mathbf{U}_{jk}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T) \qquad i, k = 1, 2, \ldots, n$$

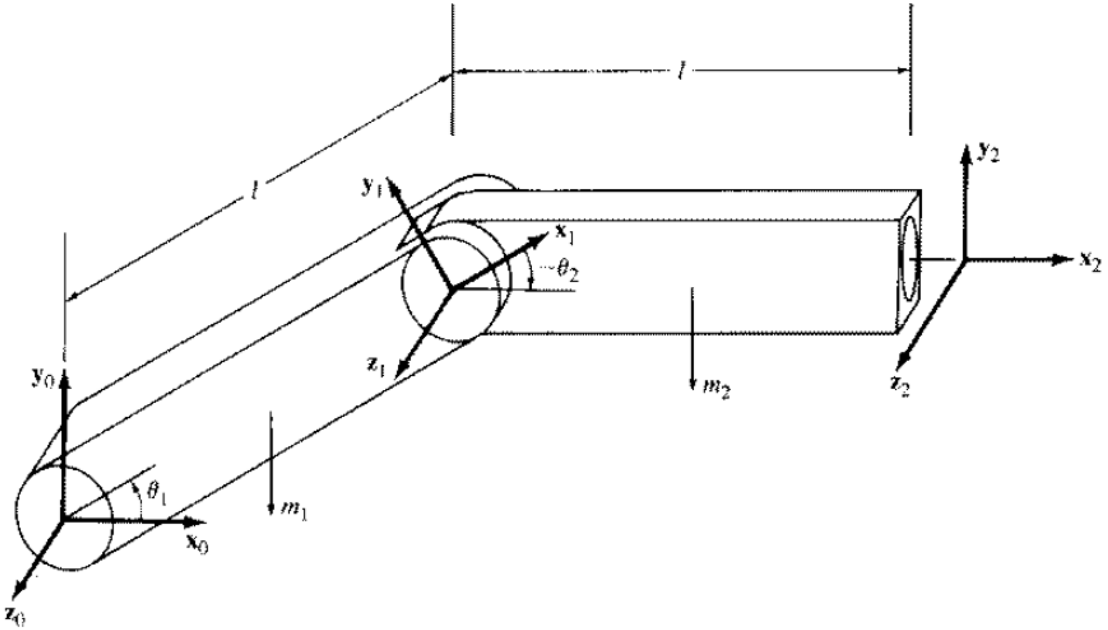
$$\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = (h_1, h_2, \ldots, h_n)^T$$

$$h_i = \sum_{k=1}^{n} \sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m \qquad i = 1, 2, \ldots, n$$

$$h_{ikm} = \sum_{j=\max(i,k,m)}^{n} \mathrm{Tr}\,(\mathbf{U}_{jkm}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T)$$

$$\mathbf{c}(\mathbf{q}) = (c_1, c_2, \ldots, c_n)^T$$

$$c_i = \sum_{j=i}^{n} (-m_j\,\mathbf{g}\,\mathbf{U}_{ji}\,{}^j\bar{\mathbf{r}}_j) \qquad i = 1, 2, \ldots, n$$



$$
{}^{0}\mathbf{A}_1 = \begin{bmatrix} C_1 & -S_1 & 0 & lC_1 \\ S_1 & C_1 & 0 & lS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}\mathbf{A}_2 = \begin{bmatrix} C_2 & -S_2 & 0 & lC_2 \\ S_2 & C_2 & 0 & lS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

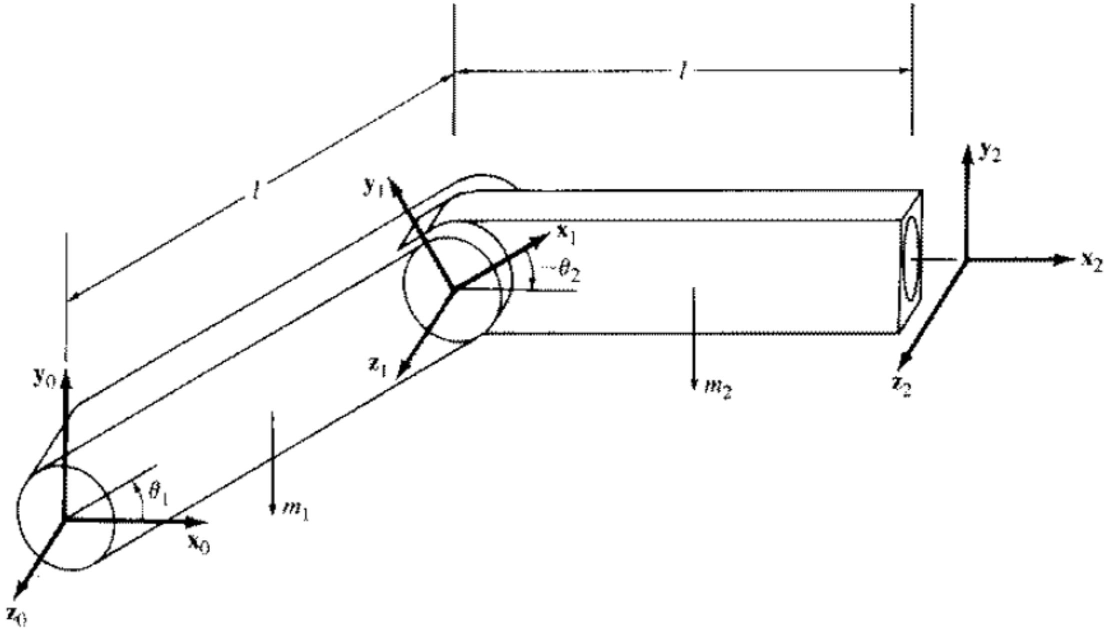
$$
{}^{0}\mathbf{A}_2 = {}^{0}\mathbf{A}_1\,{}^{1}\mathbf{A}_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l(C_{12} + C_1) \\ S_{12} & C_{12} & 0 & l(S_{12} + S_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Q}_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$\mathbf{U}_{11} = \frac{\partial\, {}^0\mathbf{A}_1}{\partial\theta_1} = \mathbf{Q}_1\, {}^0\mathbf{A}_1 = \begin{bmatrix} -S_1 & -C_1 & 0 & -lS_1 \\ C_1 & -S_1 & 0 & lC_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{U}_{21} = \frac{\partial\, {}^0\mathbf{A}_2}{\partial\theta_1} = \mathbf{Q}_1\, {}^0\mathbf{A}_2 = \begin{bmatrix} -S_{12} & -C_{12} & 0 & -l(S_{12} + S_1) \\ C_{12} & -S_{12} & 0 & l(C_{12} + C_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{U}_{22} = \frac{\partial\, {}^0\mathbf{A}_2}{\partial\theta_2} = {}^0\mathbf{A}_1 \mathbf{Q}_2\, {}^1\mathbf{A}_2 = \begin{bmatrix} -S_{12} & -C_{12} & 0 & -lS_{12} \\ C_{12} & -S_{12} & 0 & lC_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{J}_1 = \begin{bmatrix} \frac{1}{3}m_1 l^2 & 0 & 0 & -\frac{1}{2}m_1 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_1 l & 0 & 0 & m_1 \end{bmatrix} \qquad \mathbf{J}_2 = \begin{bmatrix} \frac{1}{3}m_2 l^2 & 0 & 0 & -\frac{1}{2}m_2 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_2 l & 0 & 0 & m_2 \end{bmatrix}$$



$$D_{11} = \mathrm{Tr}(\mathbf{U}_{11}\mathbf{J}_1\mathbf{U}_{11}^T) + \mathrm{Tr}(\mathbf{U}_{21}\mathbf{J}_2\mathbf{U}_{21}^T) = \tfrac{1}{3}m_1 l^2 + \tfrac{4}{3}m_2 l^2 + m_2 C_2 l^2$$

$$D_{12} = D_{21} = \mathrm{Tr}(\mathbf{U}_{22}\mathbf{J}_2\mathbf{U}_{21}^T) = m_2 l^2(-\tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{2}C_2) = \tfrac{1}{3}m_2 l^2 + \tfrac{1}{2}m_2 l^2 C_2$$

$$D_{22} = \mathrm{Tr}(\mathbf{U}_{22}\mathbf{J}_2\mathbf{U}_{22}^T) = \tfrac{1}{3}m_2 l^2 S_{12}^2 + \tfrac{1}{3}m_2 l^2 C_{12}^2 = \tfrac{1}{3}m_2 l^2$$

$$h_1 = \sum_{k=1}^{2}\sum_{m=1}^{2} h_{1km}\dot\theta_k\dot\theta_m = h_{111}\dot\theta_1^2 + h_{112}\dot\theta_1\dot\theta_2 + h_{121}\dot\theta_1\dot\theta_2 + h_{122}\dot\theta_2^2$$

$$= -\tfrac{1}{2}m_2 S_2 l^2 \dot\theta_2^2 - m_2 S_2 l^2 \dot\theta_1 \dot\theta_2$$

$$h_2 = \sum_{k=1}^{2}\sum_{m=1}^{2} h_{2km}\dot\theta_k\dot\theta_m = h_{211}\dot\theta_1^2 + h_{212}\dot\theta_1\dot\theta_2 + h_{221}\dot\theta_1 + h_{222}\dot\theta_2^2$$

$$= \tfrac{1}{2}m_2 S_2 l^2 \dot\theta_1^2$$

$$c_1 = -(m_1 \mathbf{g} \mathbf{U}_{11}\, {}^1\bar{\mathbf{r}}_1 + m_2 \mathbf{g} \mathbf{U}_{21}\, {}^2\bar{\mathbf{r}}_2) = \tfrac{1}{2} m_1 g l C_1 + \tfrac{1}{2} m_2 g l C_{12} + m_2 g l C_1$$

$$c_2 = -m_2 \mathbf{g} \mathbf{U}_{22}\, {}^2\bar{\mathbf{r}}_2 \qquad = -m_2 (\tfrac{1}{2} g l C_{12} - g l C_{12})$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{3} m_1 l^2 + \tfrac{4}{3} m_2 l^2 + m_2 C_2 l^2 & \tfrac{1}{3} m_2 l^2 + \tfrac{1}{2} m_2 l^2 C_2 \\ \tfrac{1}{3} m_2 l^2 + \tfrac{1}{2} m_2 l^2 C_2 & \tfrac{1}{3} m_2 l^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -\tfrac{1}{2} m_2 S_2 l^2 \dot{\theta}_2^2 - m_2 S_2 l^2 \dot{\theta}_1 \dot{\theta}_2 \\ \tfrac{1}{2} m_2 S_2 l^2 \dot{\theta}_1^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \tfrac{1}{2} m_1 g l C_1 + \tfrac{1}{2} m_2 g l C_{12} + m_2 g l C_1 \\ \tfrac{1}{2} m_2 g l C_{12} \end{bmatrix}$$