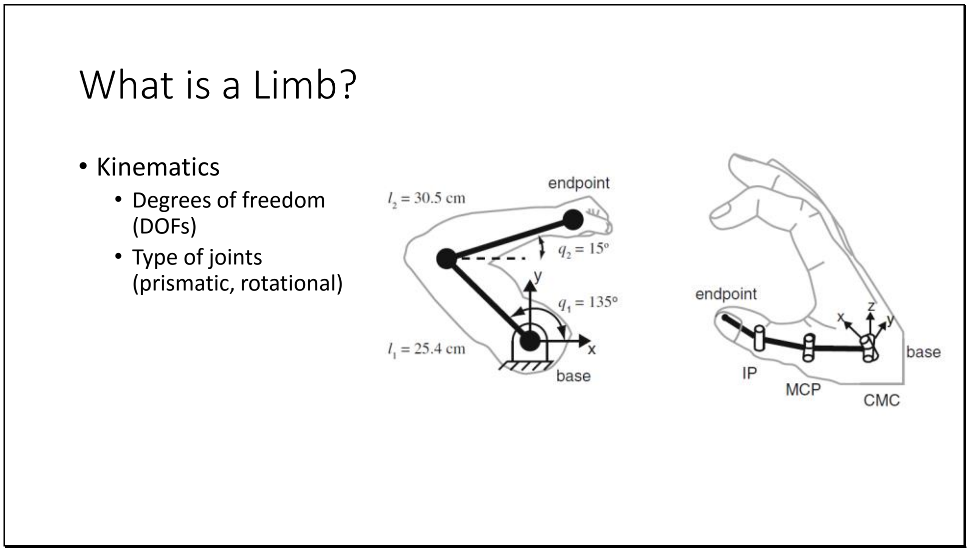
Slide 1

Kinematics Statics and Dynamics Analysis

Slide 2



Human arm model modeled as planar 2DOF serial manipulator.

Human thumb modeled as 3D serial manipulator with 5 rotational DOFs.

Some joints like knee or jaw have more complex kinematics that involve both rotation and sliding, but they are often approximated as pure rotational joints.

An important distinction between engineered and vertebrate limbs is that the former are often torque-driven, where motors act on each joint to rotate them, whereas the latter are tendon-driven, where muscles pull on tendons that act on joints by spanning 1 or more DOFs.

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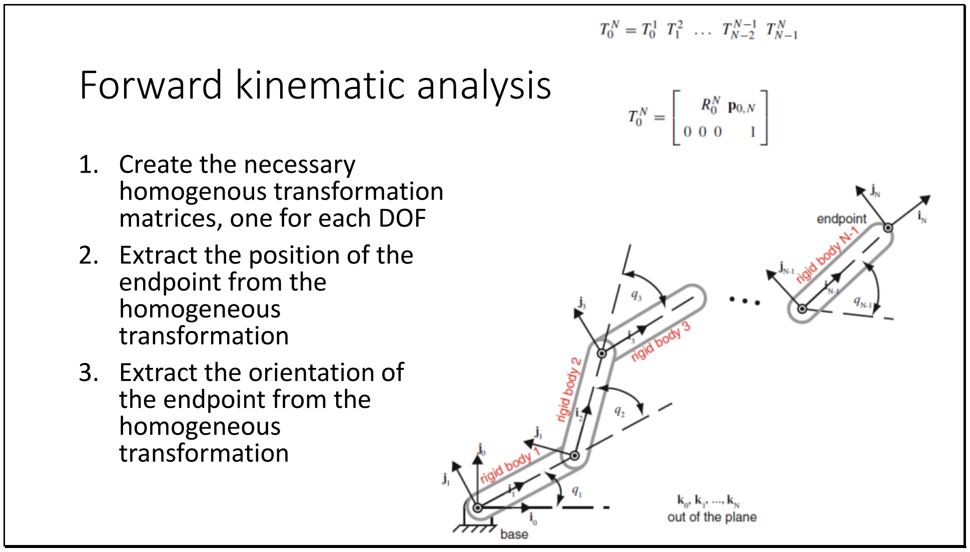

The forward kinematics of a limb determine the location and orientation of its endpoint with respect to its base, given the relative configurations of each pair of adjacent links of the limb.

This basic kinematic problem is: given a mathematical representation of the robotic or biological limb, and its joint angles and angular velocities, what is the position and velocity of its endpoint?

The posture of the limb is determined by its kinematic DOFs of the system, defined by variables $q1, q2, q3, \dots, qN$, that are also called *generalized coordinates* in mechanical analysis.

In the case where anatomical joints are assumed to be rotational joints, the generalized coordinates are angles; but they can also be linear displacements for prismatic joints in robotic systems or in anatomical joints that can slide. As mentioned above, using pure rotational joints (pin, universal, or ball-and-socket joints) is common in musculoskeletal models, but it is a critical assumption that can have important consequences to the validity and utility of the model.

- We describe rigid bodies by attaching a *frame of reference* to each body, and homogeneous transformations are used to relate adjacent frames of reference.
- the vector p0, N is the location of the endpoint with respect to the base.
- The orientation of the last link is given by the matrix R0, N. But notice that the limb is generic enough that, by having many DOFs, it raises the issue of kinematic redundancy.

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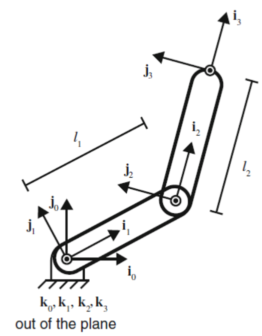
Simple Planar Limb

$$T_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^2 = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



This limb is anchored to ground, where ground is the base frame, or frame 0. The first DOF (i.e., *q*1) rotates the first link in a positive (as per the right-hand-rule) direction. The second DOF (i.e., *q*2) *rotates the second link with respect to the first link.* This is an important convention in kinematic analysis in robotics: the generalized rotational coordinates *qi* are relative to the prior body.

The first question is to define *T0,*1. It is a rotation about the first joint, where the origin of the first joint is the same as the origin of the base frame. This transformation has a rotation about the **k**0 axis of magnitude *q*1, and no translation.

c1 = cos(q1), s1 = sin(q1), c2 = cos(q2), …

It is important to point out that, even though there is no third DOF *q*3, we add the third 'dummy' transformation *T2,* 3 to place the frame of reference of the endpoint at the end of the second link.

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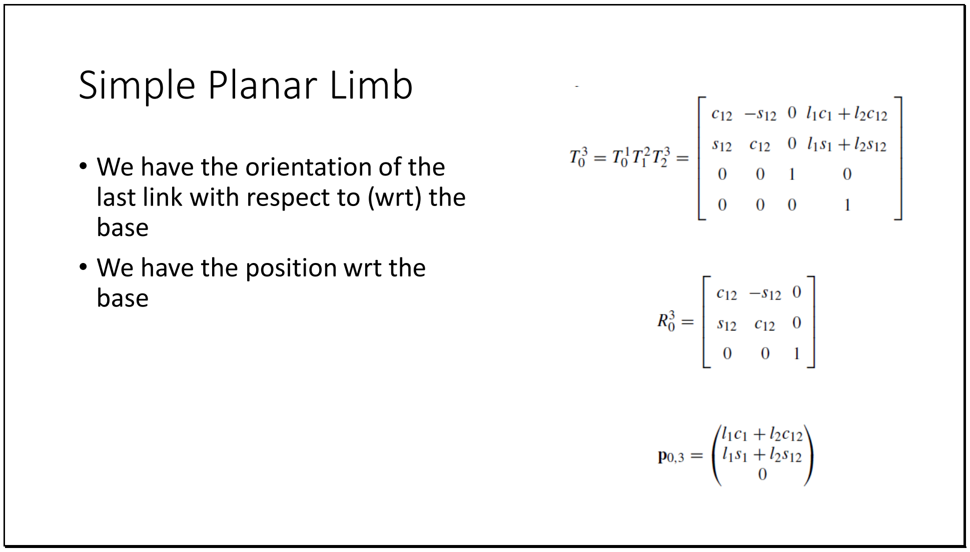

After some calculations..

c12 = cos(q1 + q2), s12 = sin(q1 + q2)

Note how the orientation and position is dependent to the general coordinates and the parameters of the link (length).

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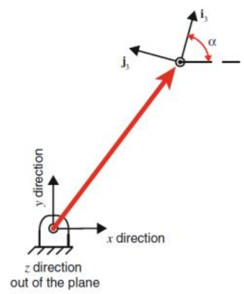
Simple Planar Limb (Geometric Model)



$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} = G(\mathbf{q}) = \begin{pmatrix} G_x(\mathbf{q}) \\ G_y(\mathbf{q}) \\ G_\alpha(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ q_1 + q_2 \end{pmatrix}$$

$$G(\mathbf{q}) = \begin{pmatrix} \text{displacement in } \mathbf{i}_0 \text{ direction} \\ \text{displacement in } \mathbf{j}_0 \text{ direction} \\ \text{rotation about the } \mathbf{k}_0 \text{ axis} \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ q_1 + q_2 \end{pmatrix}$$

Therefore, in this case the forward kinematic model (also called the geometric model), *G(***q***).* It defines the position and orientation of the end effector.

This example raises an important concept in retrospect: How many kinematic DOFs does a rigid body have on the plane? The answer is three, which are two displacements and one rotation—as revealed by the elements of *T0,* 3 and written out
explicitly in *G(***q***)*.

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How to Obtain Endpoint Velocities?

- Previously we define the geometric model (G(q))
- We want the time derivative of the forward kinematic model

$$\mathbf{x} = G(\mathbf{q})$$

$$\dot{\mathbf{x}} = \frac{G(\mathbf{q})}{dt} = \frac{\partial G(\mathbf{q})}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} = \frac{\partial G(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} \quad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} \quad \dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{pmatrix} \quad \dot{\mathbf{q}} = \begin{pmatrix} \dot{q_1} \\ \dot{q_2} \\ \vdots \\ \dot{q_N} \end{pmatrix}$$

How is the velocity of the endpoint ($\dot{\mathbf{x}}$) related to the angular velocities of the joints ($\dot{\mathbf{q}}$)? Note that a dot above a variable is a shorthand that indicates the time derivative. This is also part of the forward kinematics problem because the joint angular velocities are inputs that produce the endpoint velocities as an output.

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The Jacobian

- The J matrix is called the Jacobian of the system
- In our case $J \in R^{3xN}$
- The instantaneous 3D endpoint velocity vector can be calculated
- And when the Jacobian is **invertible** we can find the instantaneous joint angular velocities associated with a given endpoint velocity vector

$$\frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} = J(\mathbf{q}) = \begin{bmatrix} \frac{\partial G_x(\mathbf{q})}{\partial q_1} & \frac{\partial G_x(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_x(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_y(\mathbf{q})}{\partial q_1} & \frac{\partial G_y(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_y(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\alpha(\mathbf{q})}{\partial q_1} & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_N} \end{bmatrix}$$

$$\dot{x} = J(q)\dot{q}$$

$$\dot{q} = J(q)^{-1}\dot{x}$$

In some cases the Jacobian is singular and thus an inversion is not directly possible. This has a geometric meaning, but it is out of the scope of the current lecture (more advanced topics of robotics).

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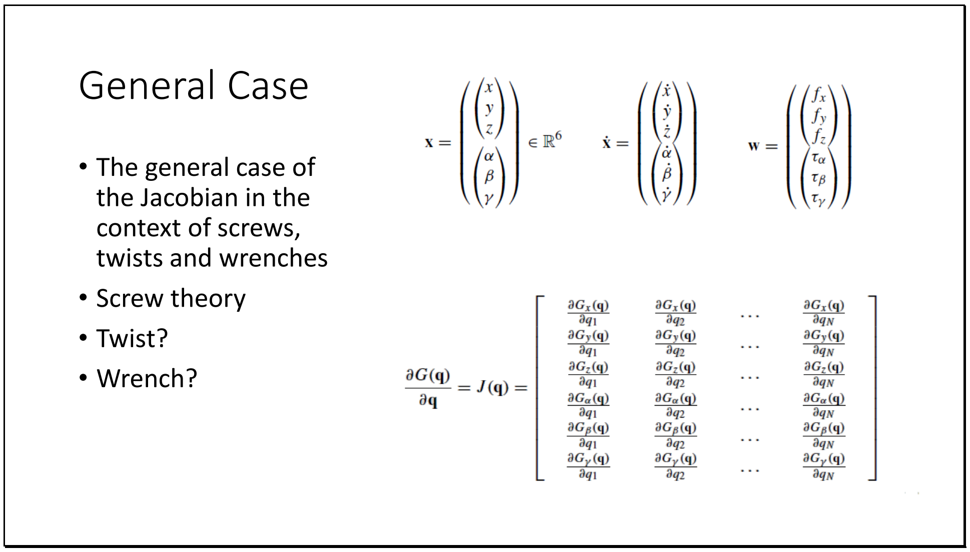

The general definition of a Jacobian needs to address the fact that the endpoint of the kinematic chain, as a rigid body, has 6 DOFs: three translations and three rotations. In the formal kinematics of rigid body mechanics, this falls within the field of screw theory. Such a combined vector is called a *screw*. It consists of a pair of 3D vectors, in this case the translations and rotations a rigid body can have—which I still call **x** because it represents the forward kinematic model of the endpoint.

In the general case where we consider all 6 DOFs of the frame of reference fixed at the endpoint, containing the 3D position and orientation vectors. The time derivative of this positional/rotational screw vector is the *twist* vector of
linear and angular velocities, respectively.

This screw concept that combines elements of different units extends to the force and torque vectors an endpoint can produce—called the endpoint *wrench* vector, where f and τ are the components of force and torque along their respective dimensions. This means that in the general case the full Jacobian has 6 rows and N columns.

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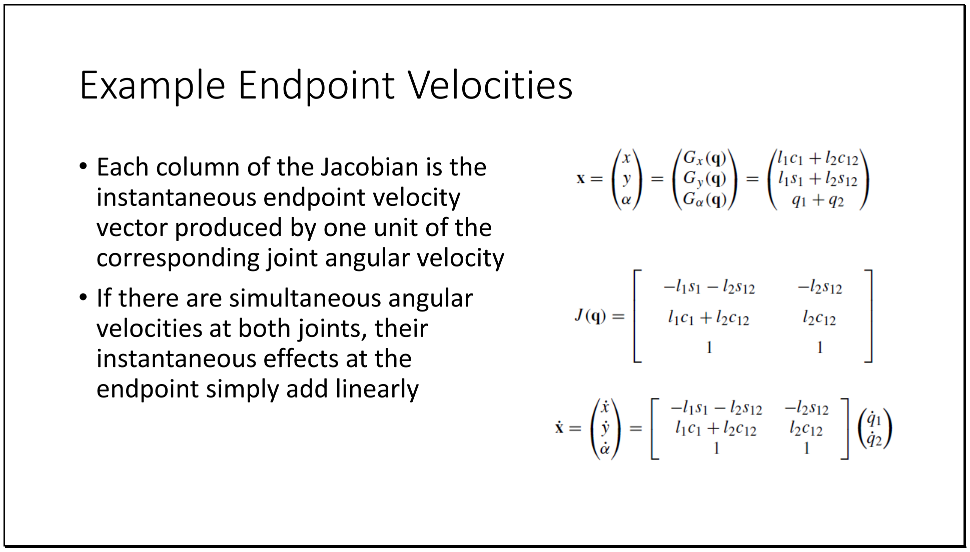

In the case of a planar 2-link, 2-joint system (previous example), the 2D forward kinematic model for the endpoint.

Each column of the Jacobian is the instantaneous endpoint velocity vector produced by one unit of the corresponding joint angular velocity (i.e., the first column of the Jacobian is the endpoint velocity vector produced by an angular velocity of 1 rad/s at the first joint if other joint angular velocities are zero, the second column is the endpoint velocity vector produced by a 1 rad/s angular velocity at the second joint if other joint angular velocities are zero, etc.).

If there are simultaneous angular velocities at both joints, their *instantaneous* effects at the endpoint simply add linearly. The last row says that the angular velocity of the endpoint is simply the sum of the angular velocity at each joint. (do the matrix multiplication)

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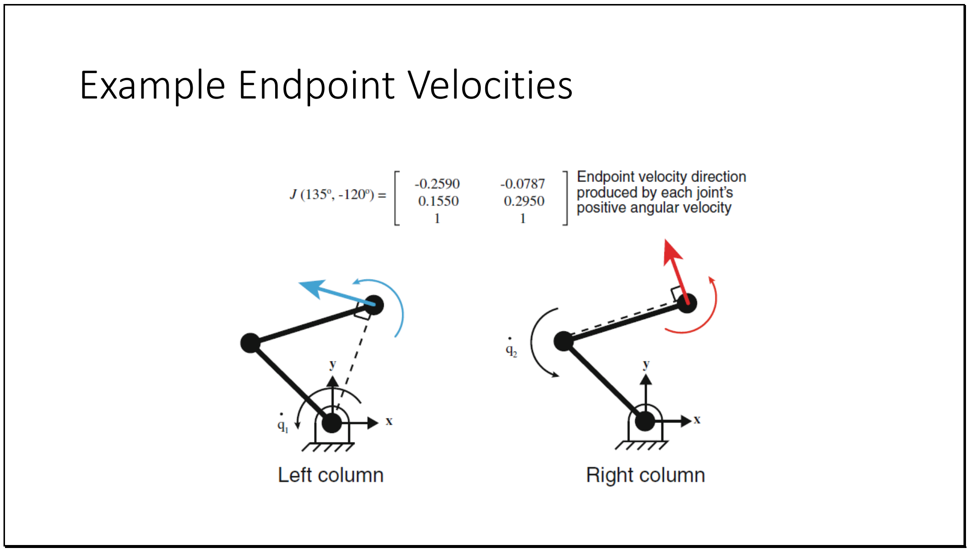

Illustration of the Jacobian for a 2 DOF planar limb. For the posture shown, the columns of the 2 × 2 Jacobian show the expected *instantaneous* endpoint linear and angular velocity for isolated angular velocities of 1 rad/s at each of the joints. If both joints are actuated, then their contribution to instantaneous endpoint velocity simply add. The limb parameters are, as per the convention in Fig. 2.3, li = 25.4cm, $l2$ = 30.5cm, $q1 = 135^\circ$, and $q2 = -120^\circ$.

First, we notice that (other than the last row) the values of the elements of the Jacobian matrix can be posture dependent (i.e., they change as the posture—or angles $q1$ and $q2$—change). Second, we are therefore forced to always speak of *instantaneous* endpoint velocities because these values only hold for that posture, and the posture is changing—by definition—given that the joints have angular velocities. And third, given that the forward kinematic model and the Jacobian involve trigonometric functions, the mapping from angular velocities to endpoint velocities changes in nonlinear ways as the motion progresses.

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Limb Mechanics

- Limb mechanics involve limb kinematics, and the forces and torques that cause limb loading and motion
- Mechanics can be both static and dynamic depending on whether motion is prevented or not, respectively
- Studying limb motions that result from applied forces and torques falls within the realm of rigid-body dynamics, which is a specialized branch of mechanics

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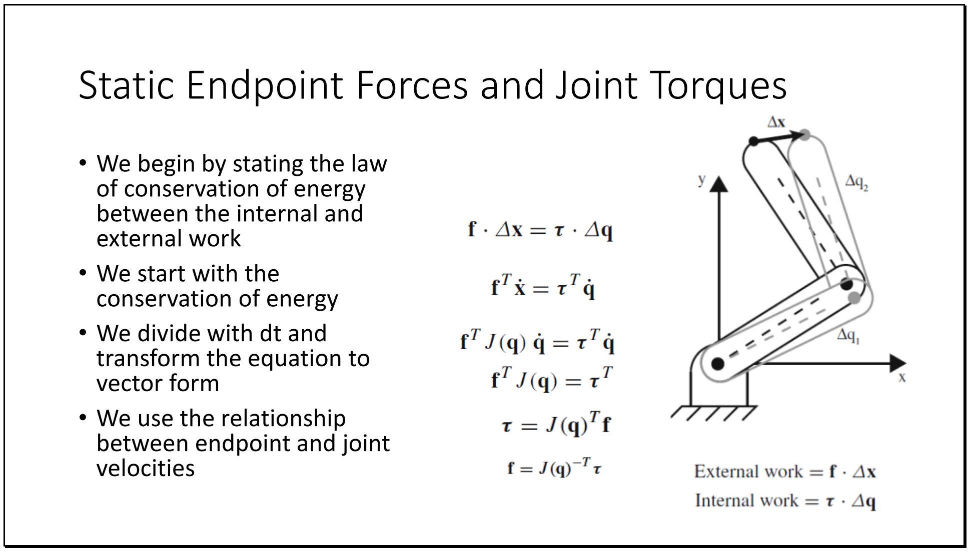

Defining the static force production capabilities of limbs begins by deriving the equation that determines the production of wrenches at the endpoint as a function of joint torques. Recall that the mechanical output of the last link of a limb can be both forces and torques (i.e., the endpoint wrench). This equation is derived using the principle of virtual work, a corollary of the law of conservation of energy.

We begin by stating the law of conservation of energy between the internal and external work that the system produces. For simplicity, I present this for a pure force at the endpoint and for torques at the joints. Recall that mechanical work is the product of force times displacement. In the linear and rotational cases, respectively.

The external work is the force vector at the endpoint **f** times the infinitesimal displacement at the endpoint $\Delta\mathbf{x} = (\Delta x, \Delta y)$. The internal work is the joint torque vector $\boldsymbol{\tau}$ times the associated infinitesimal rotation at the joints $\Delta\mathbf{q} = (\Delta q1, \Delta q2)$.

This derivation can be extended to a wrench and a screw displacement at the endpoint. The use of the dot product ensures that only forces compatible with possible displacements are considered. Similarly, the limb may have linear DOFs (like sliders) so the internal work can be from forces and displacements instead of only torques and rotations. Last, to be precise and compatible with Castigliano's theorems of virtual work, we need to speak of infinitesimal displacements (hence the term *virtual work*) so that the shape of the system (i.e., posture of the limb) does not change during the analysis.

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All Permutations of J for a Planar 2DOF Limb

$$G(\mathbf{q}) = \begin{bmatrix} l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) \\ l_2 \sin(q_1 + q_2) + l_1 \sin(q_1) \end{bmatrix}$$

$$J(\mathbf{q}) = \begin{bmatrix} -l_2 \sin(q_1 + q_2) - l_1 \sin(q_1) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) & l_2 \cos(q_1 + q_2) \end{bmatrix} \qquad \dot{x} = J(q)\dot{q}$$

$$J^T(\mathbf{q}) = \begin{bmatrix} -\sin(q_1)\, l_1 - \sin(q_1 + q_2)\, l_2 & \cos(q_1)\, l_1 + \cos(q_1 + q_2)\, l_2 \\ -\sin(q_1 + q_2)\, l_2 & \cos(q_1 + q_2)\, l_2 \end{bmatrix} \qquad \tau = J(\mathbf{q})^T \mathbf{f}$$

$$J^{-1}(\mathbf{q}) = \begin{bmatrix} -\frac{\cos(q_1 + q_2)}{l_1 \cos(q_1 + q_2) \sin(q_1) - l_1 \sin(q_1 + q_2) \cos(q_1)} & -\frac{\sin(q_1 + q_2)}{l_1 \cos(q_1 + q_2) \sin(q_1) - l_1 \sin(q_1 + q_2) \cos(q_1)} \\ \frac{l_2 \cos(q_1 + q_2) + l_1 \cos(q_1)}{l_1 l_2 \cos(q_1 + q_2) \sin(q_1) - l_1 l_2 \sin(q_1 + q_2) \cos(q_1)} & \frac{l_2 \sin(q_1 + q_2) + l_1 \sin(q_1)}{l_1 l_2 \cos(q_1 + q_2) \sin(q_1) - l_1 l_2 \sin(q_1 + q_2) \cos(q_1)} \end{bmatrix} \qquad \dot{q} = J(q)^{-1}\dot{x}$$

$$J^{-T}(\mathbf{q}) = \begin{bmatrix} \frac{\cos(q_1 + q_2)}{\cos(q_1) \sin(q_1 + q_2)\, l_1 - \sin(q_1) \cos(q_1 + q_2)\, l_1} & -\frac{\cos(q_1)\, l_1 + \cos(q_1 + q_2)\, l_2}{\cos(q_1) \sin(q_1 + q_2)\, l_1 l_2 - \sin(q_1) \cos(q_1 + q_2)\, l_1 l_2} \\ \frac{\sin(q_1 + q_2)}{\cos(q_1) \sin(q_1 + q_2)\, l_1 - \sin(q_1) \cos(q_1 + q_2)\, l_1} & -\frac{\sin(q_1)\, l_1 + \sin(q_1 + q_2)\, l_2}{\cos(q_1) \sin(q_1 + q_2)\, l_1 l_2 - \sin(q_1) \cos(q_1 + q_2)\, l_1 l_2} \end{bmatrix} \qquad \mathbf{f} = J(\mathbf{q})^{-T} \tau$$

It is important to first get a sense of what each of these forms of the Jacobian conveys. When is the Jacobian invertible, and why?

Imagine the Jacobian as a mapping from some space to another. This is what a matrix multiplication does.

- Maps from general coordinates to Cartesian coordinates
- Maps from a point force to joint torques
- Maps from Cartesian coordinates to general coordinates
- Maps from joint torques to a point force

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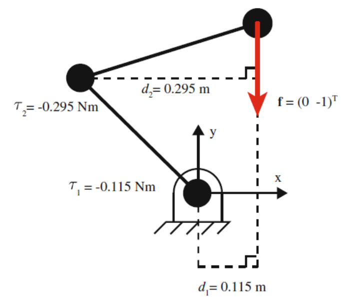
Example Static Endpoint Force

- What are the joint torques in order to exert a static 1N force in the negative y-direction?



$$\tau = J^T \mathbf{f} = \begin{bmatrix} -0.259 & 0.115 \\ -0.0787 & 0.295 \end{bmatrix} \begin{pmatrix} 0.0 \\ -1.0 \end{pmatrix} = \begin{pmatrix} -0.115 \\ -0.295 \end{pmatrix}$$

We can use Newton's first law stating that the sum of forces and torques equals zero to find joint torques needed to exert a static 1N force in the negative *y*-direction.

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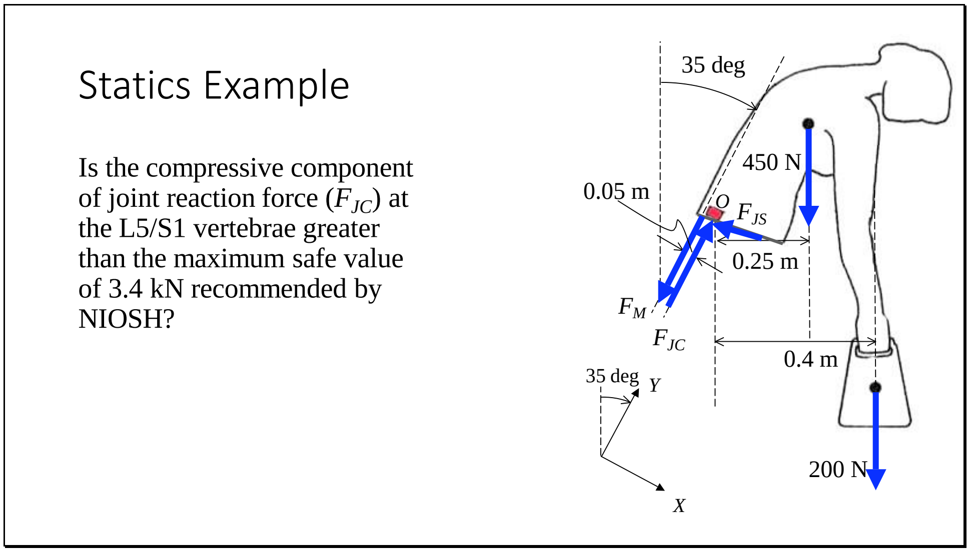


1) Compute muscle force for equilibrium

From torque equilibrium:

$Fm * 0.05 = 450 * 0.25 + 200 * 0.4$
$Fm = 3850N$

2) Project 450, 200N in the direction of Fjc

$F1 = 450 * \cos(35) = 369$
$F2 = 200 * \cos(35) = 164$

3) Add Fm and the two projected forces

$F = Fm + F1 + F2 = 4383N = Fjc$

Answer: YES, the load is beyond the safe recommended value. F_{JC} = 4380N, or approx. 6 times body weight.

Newton’s Second Law of Motion

General:

$\Sigma F_x = ma_x$ $\Sigma M_x = I\alpha_x$

$\Sigma F_y = ma_y$ $\Sigma M_y = I\alpha_y$

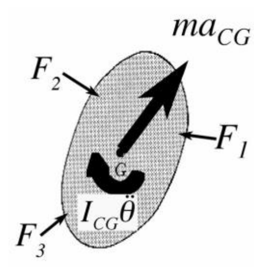
$\Sigma F_z = ma_z$ $\Sigma M_z = I\alpha_z$

2D:

$$\sum F_x = ma_{CG_x}$$

$$\sum F_y = ma_{CG_y}$$

$$\sum M_{CG} = I_{CG}\ddot{\theta}$$



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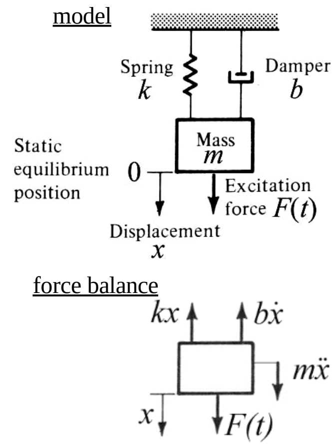
Mass-Spring-Damper Example

$$\underline{\sum F_x = m\ddot{x}:}$$

$$-b\dot{x} - kx + F(t) = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + kx = F(t)$$



In this example, we don’t have any torque, so we use the force equilibrium law.

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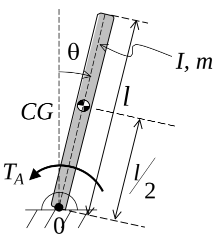
Inverted Pendulum Example

$$\underline{\Sigma M_0 = I_0\ddot{\theta}:}$$

$$mg\frac{l}{2}\sin\theta - T_A = I_0\ddot{\theta}$$

$$I_0\ddot{\theta} - mg\frac{l}{2}\sin\theta = -T_A$$



What is the torque Ta?

In this example we have only rotational movement, thus we use only the rotational law (rotational equilibrium). There are two torques: the torque due to the gravitational force of the center of mass and the active torque around 0.

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Lagrange’s Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} = \Xi_j$$

where:

ξ_j are independant generalized coordinates

Ξ_j are corresponding generalized forces

(torque if ξ_j is an angle)

$L = T - V$ (the Lagrangian function)

T = total kinetic energy of the system

V = total potential energy of the system

The Lagrange’s equations provide a general way to derive the equation of motion. Instead of forces, Lagrangian mechanics use the energies in the system.

In multibody formalization constraints can be presented so as to restrict some direction of movement. For example, the modeling of a knee joint permits rotation around an axis, but restricts all other possible movements (no translation, no rotation on any other axis). If we were to use the traditional Newton’s second law, it is very difficult to derive a compact form of the equation of motion with the least number of coordinates (free variables).

We will not get into details about how constraints are incorporated into the equation of motion, but we will focus on the concept of the Lagrange’s equations.

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} = \Xi_j \qquad (1)$$

Now *usually* we find that

$T = f(\dot{\xi}_j)$ and

$V = f(\xi_j)$

so Lagrange's equation (equation (1) above) can be written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_j}\right) + \frac{\partial V}{\partial \xi_j} = \Xi_j$$

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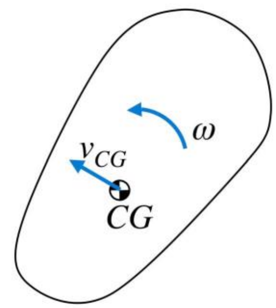
Kinetic Energy of a Rigid Body

The total kinetic energy of a rigid body that has rotational velocity $\vec{\omega}$, and linear velocity $\vec{v}_{CG}$ of its centre of gravity, is:

$$T = \frac{1}{2} m v_{CG}{}^{2} + \frac{1}{2} I_{CG} \omega^{2}$$

where m is the mass of the body, and I_{CG} is the moment of inertia about the centre of gravity.



Gravitational Potential Energy

The energy that an object possesses because of the height of its centre of mass relative to the earth's surface:

$$V(z) = mgz$$

where z = height of mass m above the earth's surface

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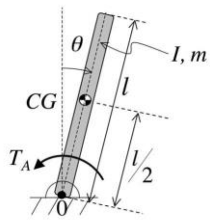
Inverted Pendulum Example



(1)Momentum

$$\Sigma M_0 = mg\frac{l}{2}\sin\theta - T_A$$

$$mg\frac{l}{2}\sin\theta - T_A = I_0\ddot{\theta}$$

$$I_0\ddot{\theta} - mg\frac{l}{2}\sin\theta = -T_A$$

(2)Energy

$$T = \frac{1}{2} I_0\dot{\theta}^2$$

$$V = mg\frac{l}{2}\cos\theta$$

$$L = T - V$$

Lagrange's Equation :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \left(\frac{\partial L}{\partial\theta}\right) = -T_A$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I_0\dot{\theta}\right) - mg\frac{l}{2}\sin\theta = -T_A$$

$$I_0\ddot{\theta} - mg\frac{l}{2}\sin\theta = -T_A$$

Same result as before!