State Space Representation

Definitions

X – State vector

- Information of the current condition of the internal variables
- N is the "dimension" of the state model (number of internal state variables)

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

$\dot{X}$ – "Next state" vector

- Derivative of the state vector
- Provides knowledge of where the states are going
  - Direction (+ or -)
  - How fast (magnitude)
- A function fo the input and the present state of the internal variables

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix}$$

State-Space Equations

General form of the state-space model

Two equations – $\dot{x}(t)$, $y(t)$

$$\dot{x}(t) = f\left(x(t), v(t), t\right)$$
$$y(t) = g\left(x(t), v(t), t\right)$$

Linear State-Space Equations

$$\dot{x}(t) = Ax(t) + Bv(t)$$
$$y(t) = Cx(t) + Dv(t)$$

$x(t),\ \dot{x}(t) \rightarrow N \times 1$ vectors

$v(t) \rightarrow R \times 1$ vector

$y(t) \rightarrow M \times 1$ vector

$A \rightarrow N \times N$ system matrix

$B \rightarrow N \times R$ input matrix

$C \rightarrow M \times N$ output matrix

$D \rightarrow M \times R$ matrix representing direct coupling from system inputs to system outputs

If A, B, C, D are constant over time, then the system is also time invariant
→ Linear Time Invariant (LTI) system

Example

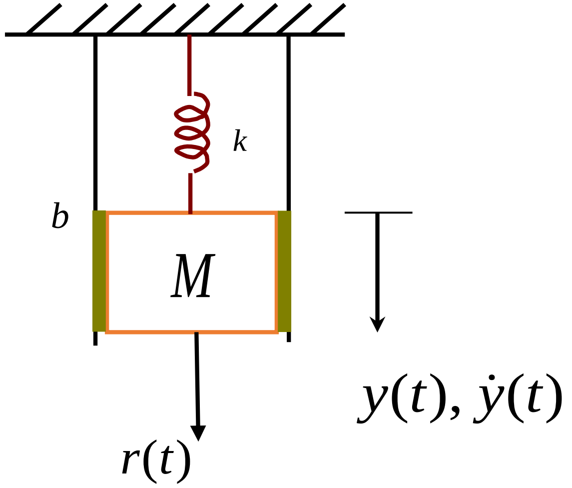
By Newton's Law

$$F = M\ddot{y}$$

$$r - ky - b\dot{y} = M\ddot{y}$$

$$\Rightarrow M\ddot{y} + b\dot{y} + ky = r$$

let $x_1 = y, x_2 = \dot{y}$



$$\Rightarrow \begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -\dfrac{b}{M}\dot{y} - \dfrac{k}{M}y + \dfrac{1}{M}r \end{cases}$$

$$= -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u \qquad (u = r)$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{M}x_1 - \dfrac{b}{M}x_2 + \dfrac{1}{M}u \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\dfrac{k}{M} & -\dfrac{b}{M} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix}}_{B} \cdot u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$