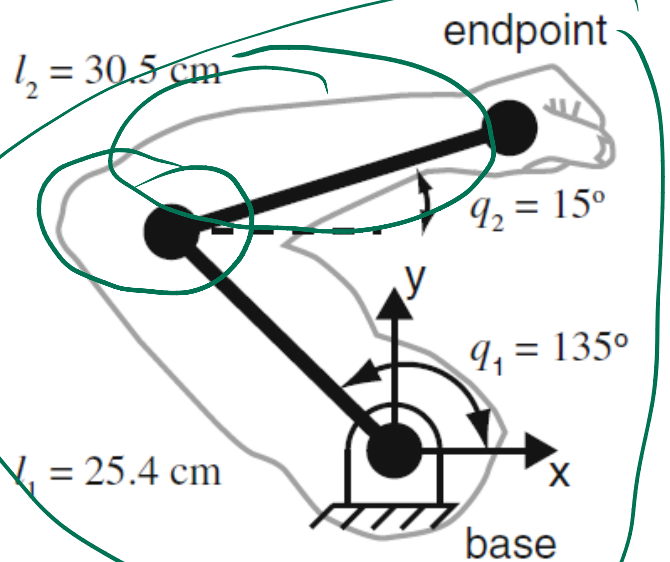
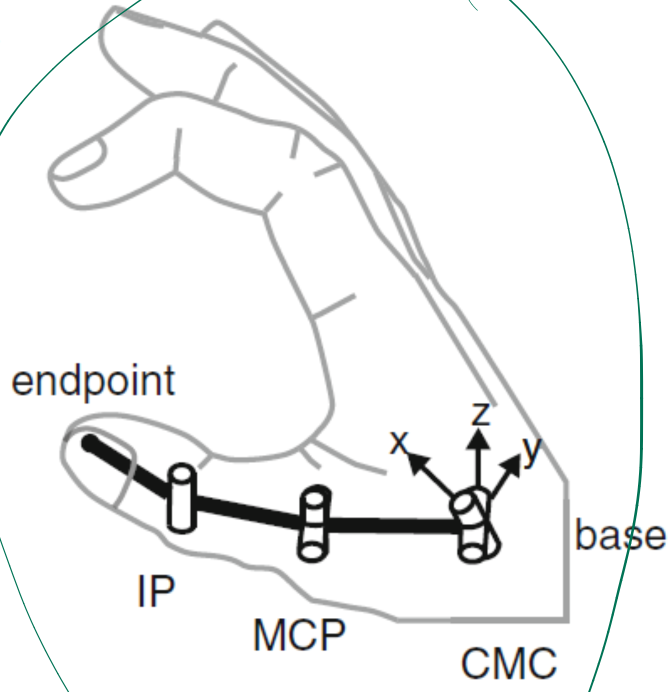
Kinematics Statics and Dynamics Analysis

What is a Limb?

- Kinematics
  - Degrees of freedom (DOFs)
  - Type of joints (prismatic, rotational)



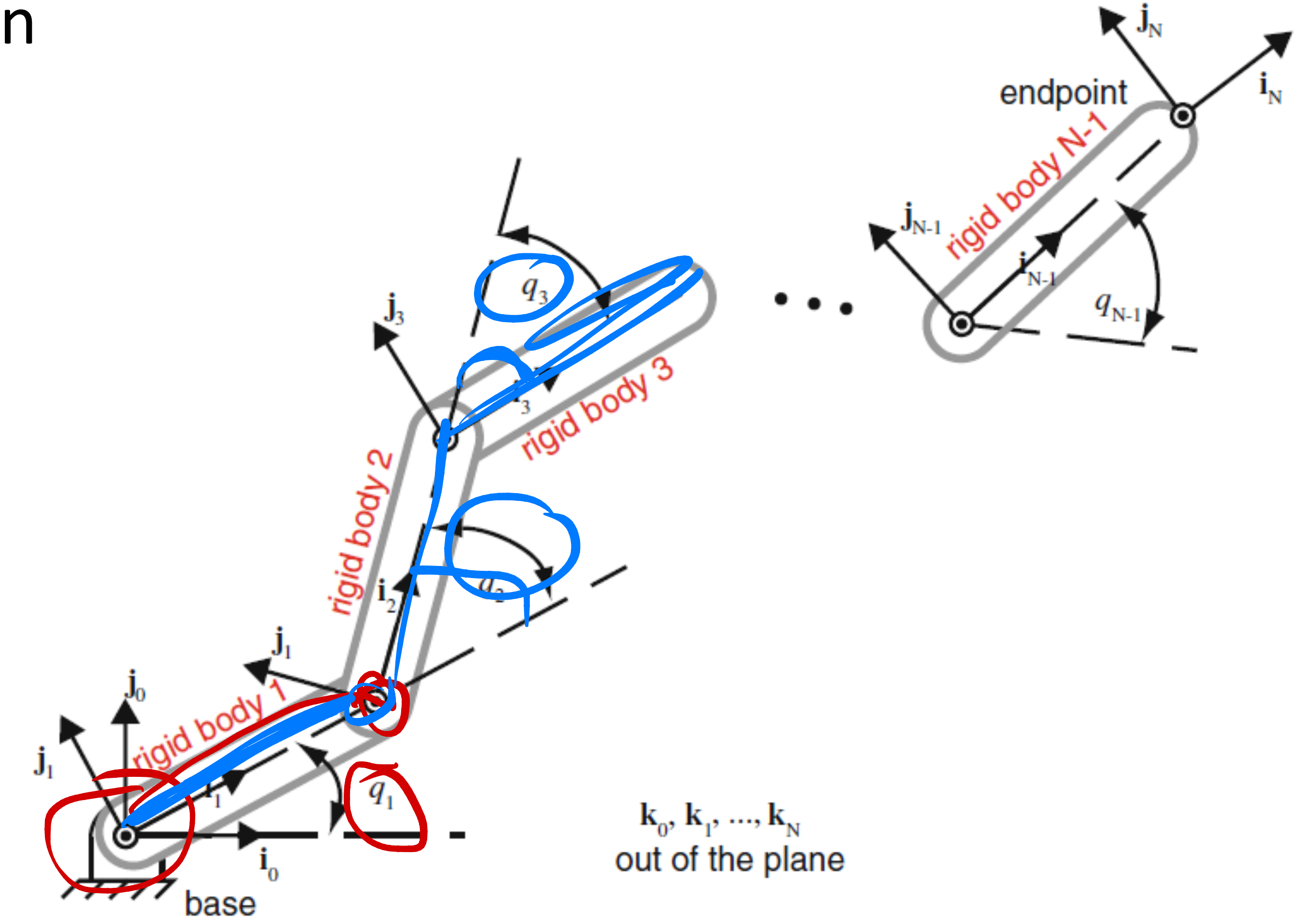


$$T_0^N = T_0^1 \; T_1^2 \; \ldots \; T_{N-2}^{N-1} \; T_{N-1}^N$$

$$T_0^N = \begin{bmatrix} R_0^N & \mathbf{p}_{0,N} \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$$

Forward kinematic analysis

1. Create the necessary homogenous transformation matrices, one for each DOF
2. Extract the position of the endpoint from the homogeneous transformation
3. Extract the orientation of the endpoint from the homogeneous transformation

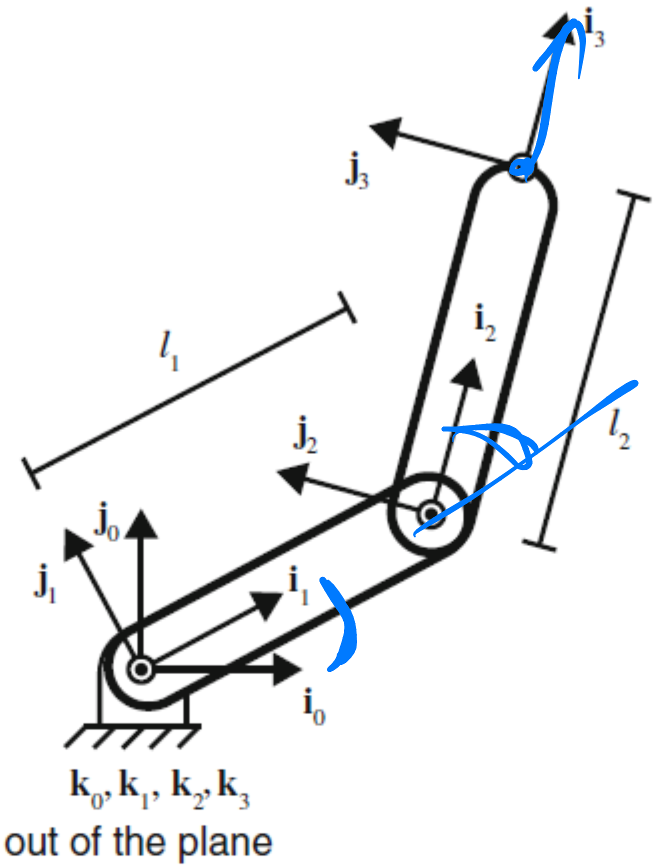


Simple Planar Limb

$$T_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^2 = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Simple Planar Limb

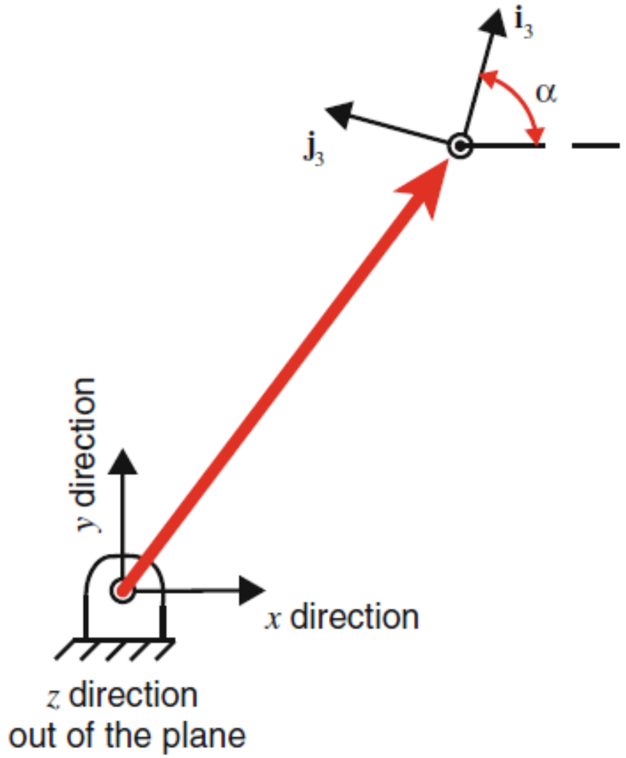
- We have the orientation of the last link with respect to (wrt) the base
- We have the position wrt the base

$$T_0^3 = T_0^1 T_1^2 T_2^3 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^3 = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

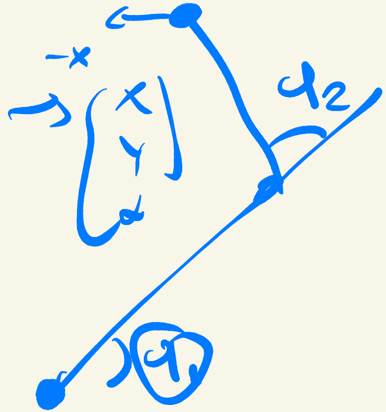
$$\mathbf{p}_{0,3} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{pmatrix}$$

Simple Planar Limb (Geometric Model)



$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} = G(\mathbf{q}) = \begin{pmatrix} G_x(\mathbf{q}) \\ G_y(\mathbf{q}) \\ G_\alpha(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ q_1 + q_2 \end{pmatrix}$$

$$G(\mathbf{q}) = \begin{pmatrix} \text{displacement in } \mathbf{i}_0 \text{ direction} \\ \text{displacement in } \mathbf{j}_0 \text{ direction} \\ \text{rotation about the } \mathbf{k}_0 \text{ axis} \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ q_1 + q_2 \end{pmatrix}$$



$$\begin{bmatrix} x \\ y \\ \alpha \end{bmatrix} = \bar{x} = G(q) \qquad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial x}{\partial q_1} & \dfrac{\partial y}{\partial q_1} & \dfrac{\partial \alpha}{\partial q_1} \\[2ex] \dfrac{\partial x}{\partial q_2} & \dfrac{\partial y}{\partial q_2} & \dfrac{\partial \alpha}{\partial q_2} \end{bmatrix}$$

How to Obtain Endpoint Velocities?

- Previously we define the geometric model (G(q))
- We want the time derivative of the forward kinematic model

$$\mathbf{x} = G(\mathbf{q})$$

$$\dot{\mathbf{x}} = \frac{G(\mathbf{q})}{dt} = \frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dt} = \frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{pmatrix}$$

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{q_1} \\ \dot{q_2} \\ \vdots \\ \dot{q_N} \end{pmatrix}$$

The Jacobian

- The J matrix is called the Jacobian of the system
- In our case $J \in R^{3xN}$
- The instantaneous 3D endpoint velocity vector can be calculated
- And when the Jacobian is **invertible** we can find the instantaneous joint angular velocities associated with a given endpoint velocity vector

$$\frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} = J(\mathbf{q}) = \begin{bmatrix} \frac{\partial G_x(\mathbf{q})}{\partial q_1} & \frac{\partial G_x(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_x(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_y(\mathbf{q})}{\partial q_1} & \frac{\partial G_y(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_y(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\alpha(\mathbf{q})}{\partial q_1} & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_N} \end{bmatrix}$$

$$\dot{x} = J(q)\dot{q}$$

$$\dot{q} = J(q)^{-1}\dot{x}$$

General Case

- The general case of the Jacobian in the context of screws, twists and wrenches
- Screw theory
- Twist?
- Wrench?

$$\mathbf{x} = \begin{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \end{pmatrix} \in \mathbb{R}^6 \qquad \dot{\mathbf{x}} = \begin{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \\ \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \\ \begin{pmatrix} \tau_\alpha \\ \tau_\beta \\ \tau_\gamma \end{pmatrix} \end{pmatrix}$$

$$\frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} = J(\mathbf{q}) = \begin{bmatrix} \frac{\partial G_x(\mathbf{q})}{\partial q_1} & \frac{\partial G_x(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_x(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_y(\mathbf{q})}{\partial q_1} & \frac{\partial G_y(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_y(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_z(\mathbf{q})}{\partial q_1} & \frac{\partial G_z(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_z(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\alpha(\mathbf{q})}{\partial q_1} & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\beta(\mathbf{q})}{\partial q_1} & \frac{\partial G_\beta(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\beta(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\gamma(\mathbf{q})}{\partial q_1} & \frac{\partial G_\gamma(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\gamma(\mathbf{q})}{\partial q_N} \end{bmatrix}$$

Example Endpoint Velocities

- Each column of the Jacobian is the instantaneous endpoint velocity vector produced by one unit of the corresponding joint angular velocity
- If there are simultaneous angular velocities at both joints, their instantaneous effects at the endpoint simply add linearly

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} = \begin{pmatrix} G_x(\mathbf{q}) \\ G_y(\mathbf{q}) \\ G_\alpha(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ q_1 + q_2 \end{pmatrix}$$

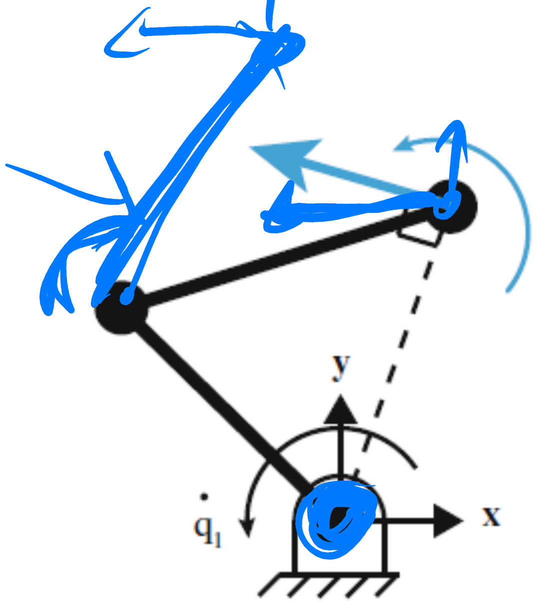
$$J(\mathbf{q}) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{pmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

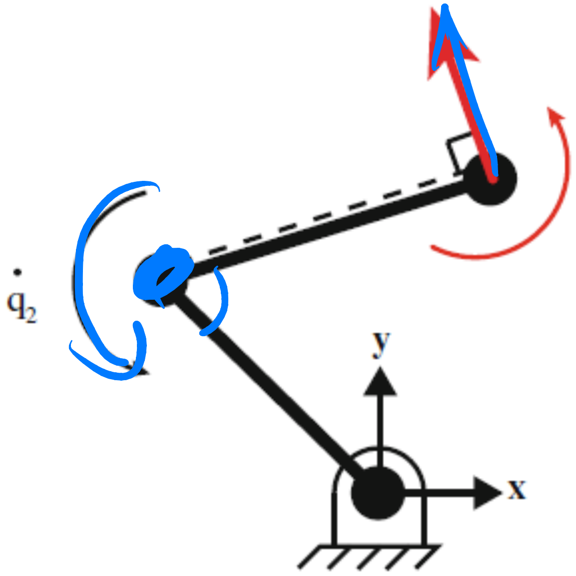
Example Endpoint Velocities

$$J(135^\circ, -120^\circ) = \begin{bmatrix} -0.2590 & -0.0787 \\ 0.1550 & 0.2950 \\ 1 & 1 \end{bmatrix}$$

Endpoint velocity direction produced by each joint's positive angular velocity



Left column



Right column

Limb Mechanics

- Limb mechanics involve limb kinematics, and the forces and torques that cause limb loading and motion
- Mechanics can be both static and dynamic depending on whether motion is prevented or not, respectively
- Studying limb motions that result from applied forces and torques falls within the realm of rigid-body dynamics, which is a specialized branch of mechanics

Static Endpoint Forces and Joint Torques

- We begin by stating the law of conservation of energy between the internal and external work
- We start with the conservation of energy
- We divide with dt and transform the equation to vector form
- We use the relationship between endpoint and joint velocities

$$\mathbf{f} \cdot \Delta\mathbf{x} = \boldsymbol{\tau} \cdot \Delta\mathbf{q}$$

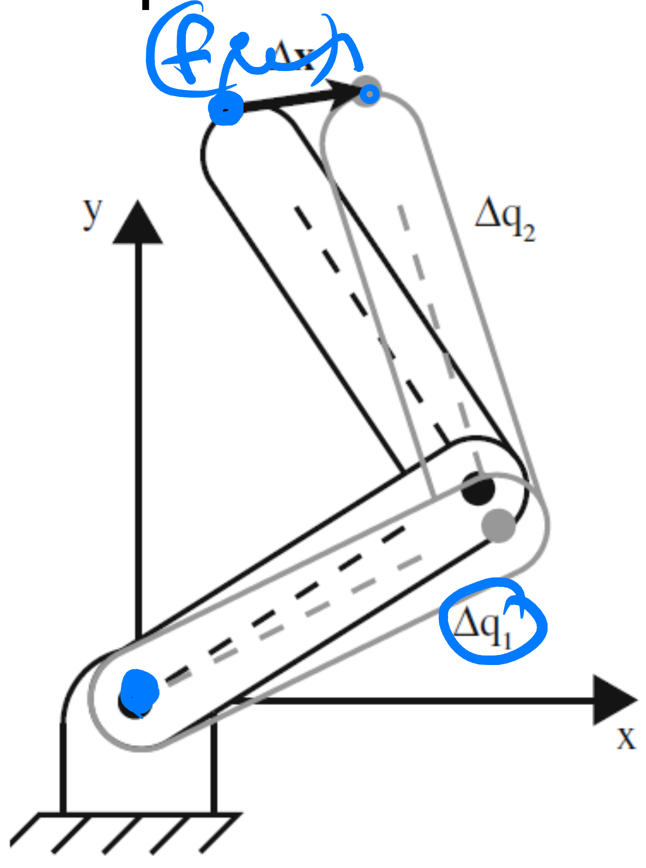
$$\mathbf{f}^T \dot{\mathbf{x}} = \boldsymbol{\tau}^T \dot{\mathbf{q}}$$

$$\mathbf{f}^T J(\mathbf{q})\, \dot{\mathbf{q}} = \boldsymbol{\tau}^T \dot{\mathbf{q}}$$

$$\mathbf{f}^T J(\mathbf{q}) = \boldsymbol{\tau}^T$$

$$\boldsymbol{\tau} = J(\mathbf{q})^T \mathbf{f}$$

$$\mathbf{f} = J(\mathbf{q})^{-T} \boldsymbol{\tau}$$



$$\text{External work} = \mathbf{f} \cdot \Delta\mathbf{x}$$

$$\text{Internal work} = \boldsymbol{\tau} \cdot \Delta\mathbf{q}$$

All Permutations of J for a Planar 2DOF Limb

$$G(\mathbf{q}) = \begin{bmatrix} l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) \\ l_2 \sin(q_1 + q_2) + l_1 \sin(q_1) \end{bmatrix}$$

$$J(\mathbf{q}) = \begin{bmatrix} -l_2 \sin(q_1 + q_2) - l_1 \sin(q_1) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) & l_2 \cos(q_1 + q_2) \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$J^T(\mathbf{q}) = \begin{bmatrix} -\sin(q_1)\, l_1 - \sin(q_1 + q_2)\, l_2 & \cos(q_1)\, l_1 + \cos(q_1 + q_2)\, l_2 \\ -\sin(q_1 + q_2)\, l_2 & \cos(q_1 + q_2)\, l_2 \end{bmatrix}$$

$$\boldsymbol{\tau} = J(\mathbf{q})^T \mathbf{f}$$

$$J^{-1}(\mathbf{q}) = \begin{bmatrix} -\dfrac{\cos(q_1 + q_2)}{l_1 \cos(q_1 + q_2) \sin(q_1) - l_1 \sin(q_1 + q_2) \cos(q_1)} & -\dfrac{\sin(q_1 + q_2)}{l_1 \cos(q_1 + q_2) \sin(q_1) - l_1 \sin(q_1 + q_2) \cos(q_1)} \\ \dfrac{l_2 \cos(q_1 + q_2) + l_1 \cos(q_1)}{l_1 l_2 \cos(q_1 + q_2) \sin(q_1) - l_1 l_2 \sin(q_1 + q_2) \cos(q_1)} & \dfrac{l_2 \sin(q_1 + q_2) + l_1 \sin(q_1)}{l_1 l_2 \cos(q_1 + q_2) \sin(q_1) - l_1 l_2 \sin(q_1 + q_2) \cos(q_1)} \end{bmatrix}$$

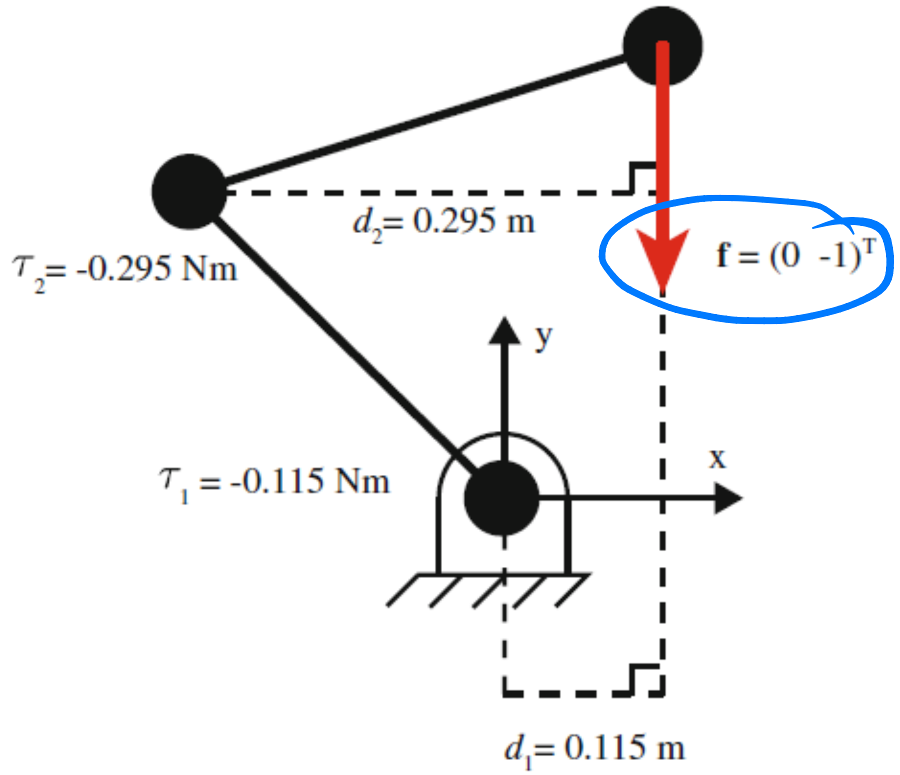
$$\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})^{-1}\dot{\boldsymbol{x}}$$

$$J^{-T}(\mathbf{q}) = \begin{bmatrix} \dfrac{\cos(q_1 + q_2)}{\cos(q_1) \sin(q_1 + q_2)\, l_1 - \sin(q_1) \cos(q_1 + q_2)\, l_1} & -\dfrac{\cos(q_1)\, l_1 + \cos(q_1 + q_2)\, l_2}{\cos(q_1) \sin(q_1 + q_2)\, l_1 l_2 - \sin(q_1) \cos(q_1 + q_2)\, l_1 l_2} \\ \dfrac{\sin(q_1 + q_2)}{\cos(q_1) \sin(q_1 + q_2)\, l_1 - \sin(q_1) \cos(q_1 + q_2)\, l_1} & -\dfrac{\sin(q_1)\, l_1 + \sin(q_1 + q_2)\, l_2}{\cos(q_1) \sin(q_1 + q_2)\, l_1 l_2 - \sin(q_1) \cos(q_1 + q_2)\, l_1 l_2} \end{bmatrix}$$

$$\mathbf{f} = J(\mathbf{q})^{-T} \boldsymbol{\tau}$$

Example Static Endpoint Force

- What are the joint torques in order to exert a static 1N force in the negative y-direction?

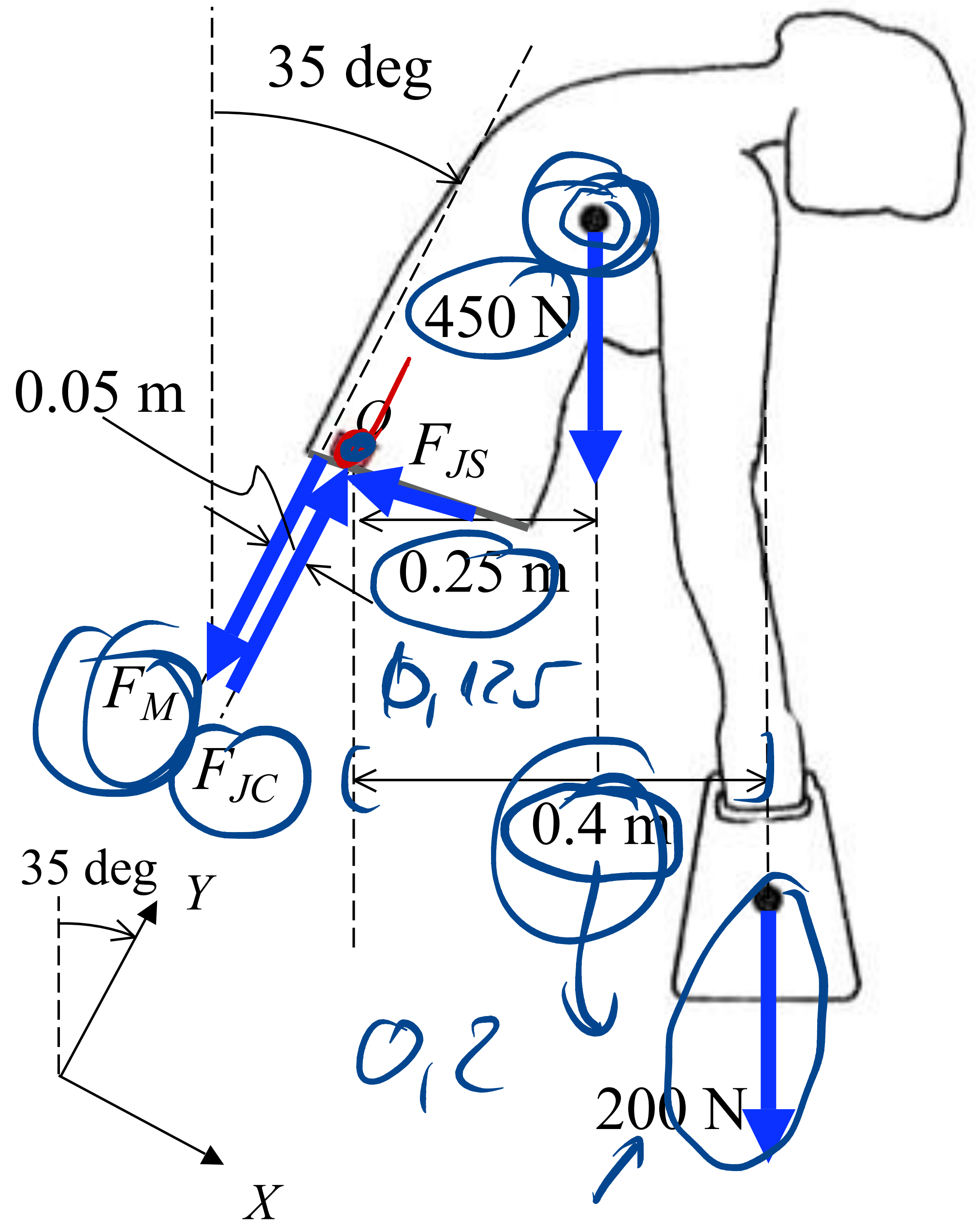


$$\tau = J^T \mathbf{f} = \begin{bmatrix} -0.259 & 0.115 \\ -0.0787 & 0.295 \end{bmatrix} \begin{pmatrix} 0.0 \\ -1.0 \end{pmatrix} = \begin{pmatrix} -0.115 \\ -0.295 \end{pmatrix}$$

Statics Example

Is the compressive component of joint reaction force (F_{JC}) at the L5/S1 vertebrae greater than the maximum safe value of 3.4 kN recommended by NIOSH?

$F_m \cdot 0{,}05 = 450 \cdot 0{,}25 + 200 \cdot 0{,}4$



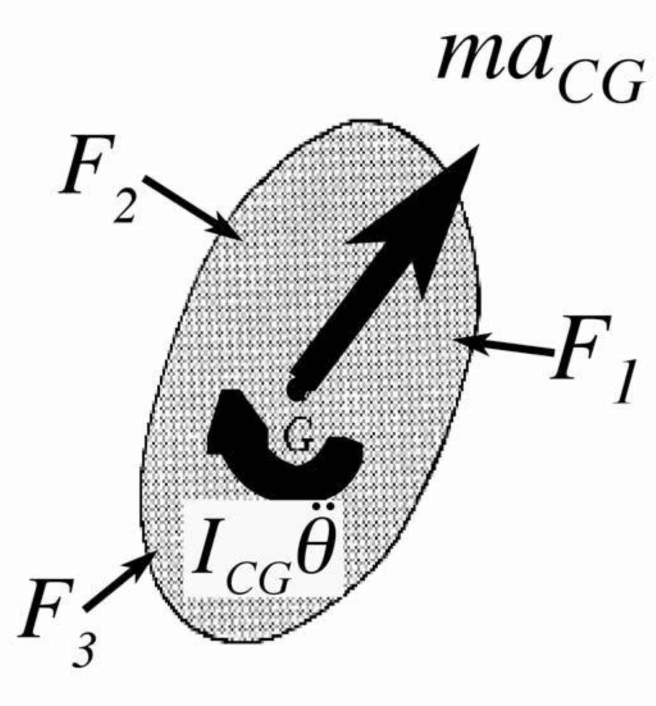
Newton's Second Law of Motion

General:

$$\Sigma F_x = ma_x \qquad \Sigma M_x = I\alpha_x$$
$$\Sigma F_y = ma_y \qquad \Sigma M_y = I\alpha_y$$
$$\Sigma F_z = ma_z \qquad \Sigma M_z = I\alpha_z$$

2D :

$$\sum F_x = ma_{CG_x}$$
$$\sum F_y = ma_{CG_y}$$
$$\sum M_{CG} = I_{CG}\ddot{\theta}$$



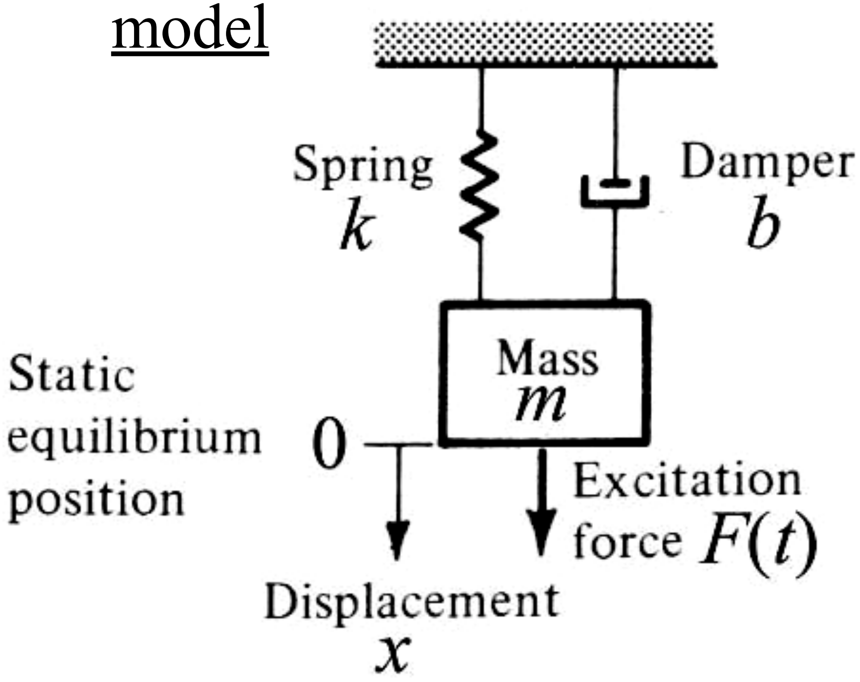
Mass-Spring-Damper Example

$$\underline{\sum F_x = m\ddot{x}:}$$

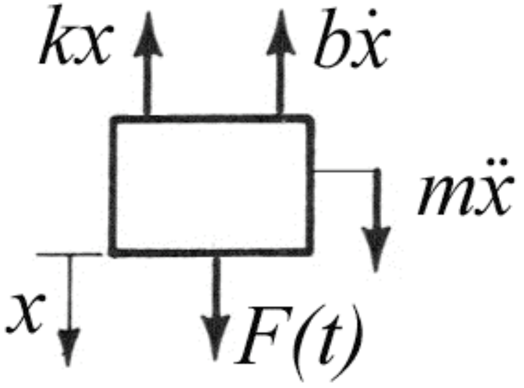
$$-b\dot{x} - kx + F(t) = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

model



force balance

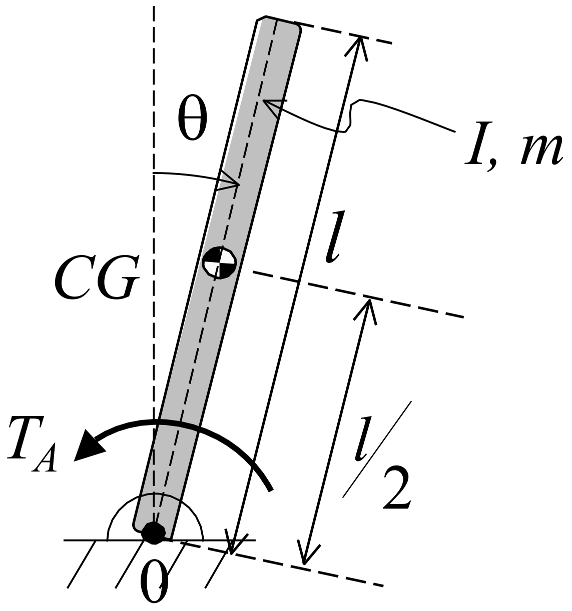


Inverted Pendulum Example

$$\underline{\Sigma M_0 = I_0\ddot{\theta}:}$$

$$mg\frac{l}{2}\sin\theta - T_A = I_0\ddot{\theta}$$

$$I_0\ddot{\theta} - mg\frac{l}{2}\sin\theta = -T_A$$



Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} = \Xi_j$$

where :

ξ_j are independant generalized coordinates

Ξ_j are corresponding generalized forces

(torque if ξ_j is an angle)

$L = T - V$ (the Lagrangian function)

T = total kinetic energy of the system

V = total potential energy of the system

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} = \Xi_j \qquad (1)$$

Now *usually* we find that

$T = f\left(\dot{\xi}_j\right)$ and

$V = f\left(\xi_j\right)$

so Lagrange's equation (equation (1) above)

can be written as:

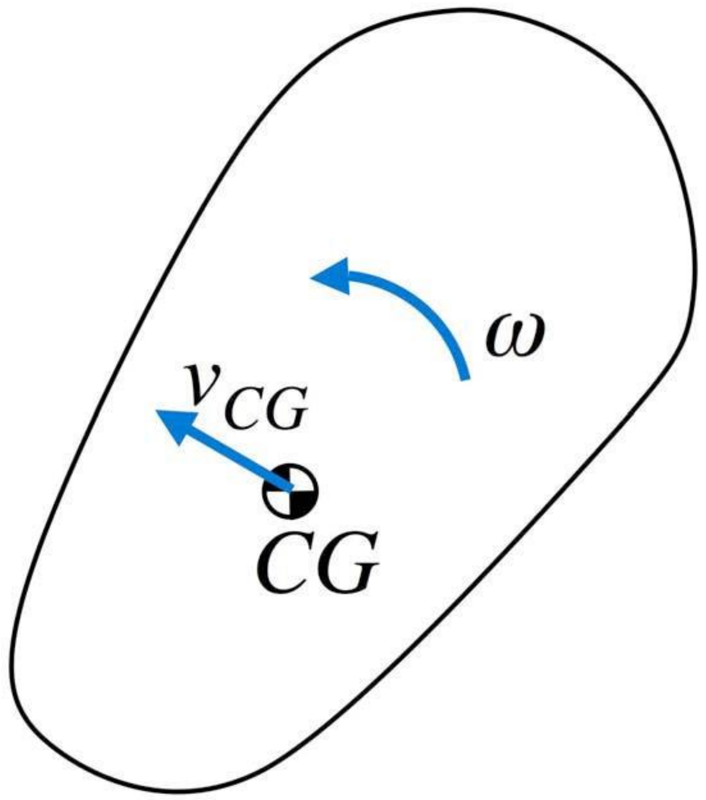
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_j}\right) + \frac{\partial V}{\partial \xi_j} = \Xi_j$$

Kinetic Energy of a Rigid Body

The total kinetic energy of a rigid body that has rotational velocity $\vec{\omega}$, and linear velocity $\vec{v}_{CG}$ of its centre of gravity, is :

$$T = \frac{1}{2} m v_{CG}^{\;2} + \frac{1}{2} I_{CG} \omega^2$$

where m is the mass of the body, and I_{CG} is the moment of inertia about the centre of gravity.



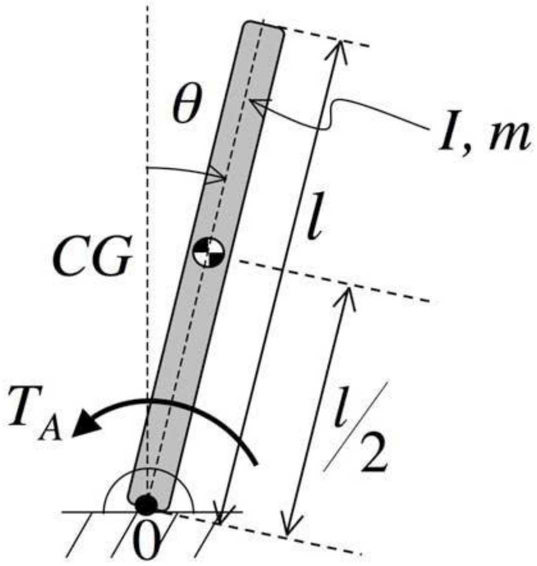
Gravitational Potential Energy

The energy that an object possesses because of the height of its centre of mass relative to the earth's surface:

$$V(z) = mgz$$

where z = height of mass m above the earth's surface

Inverted Pendulum Example



(1)Momentum

$$\Sigma M_0 = mg\frac{l}{2}\sin\theta - T_A$$

$$mg\frac{l}{2}\sin\theta - T_A = I_0\ddot{\theta}$$

$$I_0\ddot{\theta} - mg\frac{l}{2}\sin\theta = -T_A$$

(2)Energy

$$T = {}^1\!/_2\, I_0\dot{\theta}^2$$

$$V = mg\frac{l}{2}\cos\theta$$

$$L = T - V$$

Lagrange's Equation :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \left(\frac{\partial L}{\partial\theta}\right) = -T_A$$

$$\frac{d}{dt}\left(I_0\dot{\theta}\right) - mg\frac{l}{2}\sin\theta = -T_A$$

$$I_0\ddot{\theta} - mg\frac{l}{2}\sin\theta = -T_A$$