Method of moments estimators

$$\bar{X} = \frac{\sum x_i}{n} = E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$\frac{\sum x_i^2}{n} = E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

$$\frac{\sum x_i^3}{n} = E(X^3) = \int_{-\infty}^{+\infty} x^3 f(x)\,dx$$

solving moments estimators ⟸ p equations with p unknown parameters

↳ the number of the parameters of $f(x)$

Maximum likelihood estimation

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

log-likelihood

$$\ell(\theta) = \log L(\theta) = \log \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \sum \log \prod f(x_i;\theta)$$

$$\frac{\partial \ell(\theta)}{\partial \theta} = 0 \rightarrow \text{solution} \rightarrow \text{MLE } \hat{\theta} \text{ of}$$

Example

known, fixed

random sample

$X_1, X_2, \ldots, X_n \sim Gamma(r, \lambda)$

so we have only one parameter, λ

MLE $\hat{\lambda}$ of λ

likelihood

$$L(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda)$$

$$= \prod \frac{\lambda^r x_i^{r-1} e^{-\lambda x_i}}{\Gamma(r)}$$

$\Gamma(a)$ complete gamma function

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-\lambda x} dx$$

$$\ell(\lambda) = \log L(\lambda) = \sum \log \frac{\lambda^r x_i^{r-1} e^{-\lambda x_i}}{\Gamma(r)}$$

$$= \sum \left(r\log\lambda + (r-1)\log x_i - \lambda x_i - \log\Gamma(r) \right)$$

$$l(\lambda) = \sum \left(r \log \lambda + (r-1) \log x_i - \lambda x_i - \log \Gamma(r) \right)$$

$$\frac{\partial l(\lambda)}{\partial \lambda} = \sum_{i=1}^{n} \left(\frac{r}{\lambda} + 0 - x_i - 0 \right)$$

$$= n \frac{r}{\lambda} - \sum x_i$$

$$\frac{\partial l(\lambda)}{\partial \lambda} = 0 \implies \frac{nr}{\hat{\lambda}} - \sum x_i = 0$$

$$\frac{nr}{\lambda} = \sum x_i$$

$$\hat{\lambda} = \frac{nr}{\sum x_i} = \frac{r}{\sum x_i / n} = \frac{r}{\bar{x}}$$

$$\boxed{\hat{\lambda} = \frac{r}{\bar{x}}} \rightarrow \text{MLE}$$

least squares estimator

~~example~~

$X_1, \dots, X_n \sim \exp(\lambda)$, $\lambda > 0$

$f(x;\lambda) = \lambda e^{-\lambda x}$ $x > 0$

$E(X) = \int_0^{+\infty} x f(x) dx =$

$= \dots = \frac{1}{\lambda}$

$$S = \sum_{i=1}^{n} \left(x_i - \frac{1}{\lambda} \right)^2$$

$\hookrightarrow E(X)$

$$\frac{\partial S}{\partial \lambda} = \sum_{i=1}^{n} 2\left(x_i - \frac{1}{\lambda} \right)\left(-\frac{1}{\lambda} \right)'$$

$$= \sum_{i=1}^{n} 2\left(x_i - \frac{1}{\lambda} \right) \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} \sum \left(x_i - \frac{1}{\lambda} \right)$$

$$\frac{\partial S}{\partial \lambda} = 0 \Rightarrow \sum \left(x_i - \frac{1}{\lambda} \right) = 0$$

$$\Sigma x_i - \frac{n}{\lambda} = 0$$

$$\Sigma x_i = \frac{n}{\lambda} \Rightarrow \hat{\lambda} = \frac{n}{\Sigma x_i}$$

$$\boxed{\hat{\lambda} = 1/\bar{x}}$$