Lecture 1: Introduction to Signals Notes

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1 Signals and Their Taxonomy

1.1 What Is a Signal?

In engineering and science, a *signal* is the primary mathematical tool used to represent information about the world. Informally, a signal is anything that *varies* and carries information about a physical phenomenon, a system, or a process.

A **signal** is a *function* that represents information.

This definition is intentionally broad. What matters is not the physical origin of the signal, but the fact that it can be described as a function.

Mathematically, we denote a signal as

$$x(\cdot),$$

emphasizing that it is a mapping from some input variable (or variables) to a value. In most applications, the independent variable is time, leading to the familiar notations

$$x(t) \quad \text{(continuous-time signal)}, \qquad x[n] \quad \text{(discrete-time signal)}.$$

Examples.

- The voltage across a resistor as a function of time
- An audio waveform representing air pressure variations
- Temperature measurements from a sensor recorded every second
- Pixel intensity values along a row of an image

Although these examples come from very different domains (electronics, acoustics, sensing, vision), they are all treated mathematically as signals.

1.2 Signals as Mathematical Mappings

To make the concept precise, a signal can be defined as a mapping between two sets:

$$\boldsymbol{x} : \mathcal{I} \longrightarrow \mathcal{O}.$$

Here:

- $\mathcal{I}$ is the **input domain**, describing how the signal is indexed
- $\mathcal{O}$ is the **output space**, describing the type of values the signal takes

This viewpoint is fundamental, because many signal classifications arise naturally from different choices of $\mathcal{I}$ and $\mathcal{O}$.

Input Domain

The input domain specifies the variables with respect to which the signal is defined:

- $\mathbb{R}$ or $\mathbb{R}^N$: continuous variables (e.g. time, space, space–time)
- $\mathbb{Z}$ or $\mathbb{Z}^N$: discrete variables (e.g. sample index, pixel grid)
- Mixed continuous–discrete domains

If $N > 1$, the signal is called *multidimensional*. Typical examples include:

- Images: $x[m, n]$ with $(m, n) \in \mathbb{Z}^2$
- Videos: $x[m, n, t]$ combining space and time
- Spatial sensor fields (e.g. temperature over an area)

Output Space

The output space determines the nature of the signal values:

- $\mathbb{R}$: real-valued signals (most physical measurements)
- $\mathbb{C}$: complex-valued signals (e.g. Fourier-domain representations)
- $\mathbb{R}^M$ or $\mathbb{C}^M$: vector-valued or multichannel signals
- Finite sets: categorical or symbolic signals (e.g. digital symbols, logical values)

Examples.

- Grayscale image: $M = 1$
- RGB image: $M = 3$
- IMU measurements (accelerometer + gyroscope): $M = 6$

1.3 Domain vs Codomain: Signal Taxonomy

We can classify signals based on two fundamental aspects:

- **what indexes the signal** (domain)
- **what values it takes** (codomain)

Domain examples. Time, space, pixel coordinates, frequency, sample index, frame number.

Codomain examples. Real numbers, complex numbers, vectors, symbols, logical values.

Important remark. Many distinctions that appear profound in applications are mathematically simple once the domain and codomain are clearly specified.

Continuous vs Discrete Time and Amplitude

A fundamental classification of signals concerns whether the domain (input/time) and codomain (output/amplitude) are continuous or discrete.

Time (input).

- **Continuous-time**: defined for all $t \in \mathbb{R}$
- **Discrete-time**: defined only at integer indices $n \in \mathbb{Z}$

Amplitude (output).

- **Continuous amplitude**: values in $\mathbb{R}$ or $\mathbb{C}$
- **Discrete amplitude**: values from a finite or countable set

Combining these leads to four important cases:

- **Analog signals**: continuous time, continuous amplitude
- **Discrete-time signals**: discrete time, continuous amplitude
- **Quantized signals**: continuous time, discrete amplitude
- **Digital signals**: discrete time, discrete amplitude

Sampling discretizes time, while quantization discretizes amplitude. Digital signal processing typically assumes that both steps have already occurred.

1.4 Periodic and Aperiodic Signals

A signal is called *periodic* if it repeats itself exactly after a fixed interval.

Continuous-time periodicity. A continuous-time signal $x(t)$ is periodic if there exists $T > 0$ such that

$$x(t) = x(t + T) \quad \forall t.$$

Discrete-time periodicity. A discrete-time signal $x[n]$ is periodic if there exists an integer $N > 0$ such that

$$x[n] = x[n + N] \quad \forall n.$$

If no such T or N exists, the signal is called *aperiodic*.

Example. The sinusoid

$$x(t) = \cos(2\pi f_0 t)$$

is periodic with period $T = 1/f_0$, whereas a finite-duration pulse is aperiodic.

1.4.1 Sum of Periodic Signals

So far we have defined periodicity for a single signal. A very important practical question is:

> If we add two periodic signals, is the result still periodic?

The answer is subtle: sometimes yes, sometimes no, depending on how the periods (or frequencies) are related.

Continuous-Time Case

Setup. Let

$$x_1(t) \text{ be periodic with period } T_1, \qquad x_2(t) \text{ be periodic with period } T_2.$$

This means:

$$x_1(t+T_1) = x_1(t), \qquad x_2(t+T_2) = x_2(t), \qquad \forall t.$$

Now define their sum:

$$x(t) = x_1(t) + x_2(t).$$

When is the sum periodic?

The sum $x(t)$ is periodic if there exists some $T > 0$ such that

$$x(t+T) = x(t) \quad \forall t.$$

Compute:

$$x(t+T) = x_1(t+T) + x_2(t+T).$$

For this to equal $x(t)$, we need:

$$x_1(t+T) = x_1(t) \quad \text{and} \quad x_2(t+T) = x_2(t).$$

Thus T must be a common period of both signals:

$$T = k_1 T_1 = k_2 T_2$$

for some integers $k_1, k_2 \in \mathbb{Z}$.

This is possible if and only if the ratio of the periods is rational:

$$\boxed{x_1(t) + x_2(t) \text{ is periodic} \quad \Longleftrightarrow \quad \frac{T_1}{T_2} \in \mathbb{Q}.}$$

Fundamental period. When the ratio is rational, the period of the sum is the smallest positive common multiple:

$$T = \operatorname{lcm}(T_1, T_2).$$

Example: Aperiodic sum

Consider:

$$x(t) = \cos(2\pi t) + \cos\Big(2\pi\sqrt{2}\,t\Big).$$

The first cosine has frequency $f_1 = 1$, hence

$$T_1 = \frac{1}{f_1} = 1.$$

The second cosine has frequency $f_2 = \sqrt{2}$, hence

$$T_2 = \frac{1}{f_2} = \frac{1}{\sqrt{2}}.$$

Now the ratio is:

$$\frac{T_1}{T_2} = \frac{1}{1/\sqrt{2}} = \sqrt{2}.$$

Since

$$\sqrt{2} \notin \mathbb{Q},$$

there is no common period. Therefore:

$$\boxed{x(t) \text{ is not periodic.}}$$

Interpretation. The two oscillations never "line up" again exactly, so the waveform never repeats perfectly.

Discrete-Time Case

The discrete-time situation is different, because periodicity depends on integer index shifts.

Single sinusoid periodicity. Consider:

$$x[n] = \cos(\Omega n).$$

This signal is periodic if there exists an integer $N > 0$ such that:

$$x[n+N] = x[n].$$

Using the cosine periodicity:

$$\cos(\Omega(n+N)) = \cos(\Omega n) \quad \iff \quad \Omega N = 2\pi k$$

for some integer k.

Thus:

$$\boxed{x[n] \text{ is periodic} \quad \iff \quad \frac{\Omega}{2\pi} \in \mathbb{Q}.}$$

Sum of discrete-time sinusoids

Let:

$$x_1[n] = \cos(\Omega_1 n), \qquad x_2[n] = \cos(\Omega_2 n).$$

Each signal is periodic if:

$$\frac{\Omega_1}{2\pi} \in \mathbb{Q}, \qquad \frac{\Omega_2}{2\pi} \in \mathbb{Q}.$$

Now consider the sum:

$$x[n] = x_1[n] + x_2[n].$$

Periodicity condition. The sum is periodic if both signals are periodic and their periods share a common multiple:

$$\boxed{x_1[n] + x_2[n] \text{ is periodic if both are periodic and their periods have a common multiple.}}$$

Equivalently, there must exist an integer N such that:

$$\Omega_1 N = 2\pi k_1, \qquad \Omega_2 N = 2\pi k_2$$

for integers k_1, k_2.

Key difference from continuous time

A striking fact is:

> Discrete-time sinusoids are periodic for many more frequencies than continuous-time sinusoids.

Because in discrete time, periodicity depends on whether $\Omega/2\pi$ is rational, not on the absolute value of Ω.

Summary

- In continuous time, the sum of two periodic signals is periodic only if their periods are commensurate:
$$\frac{T_1}{T_2} \in \mathbb{Q}.$$
- In discrete time, a sinusoid is periodic if:
$$\frac{\Omega}{2\pi} \in \mathbb{Q}.$$
- A sum of periodic DT signals is periodic if they share a common period.

This topic becomes especially important in Fourier analysis, where signals are often decomposed into sums of sinusoids.

1.5 Signal Support: Causality

The *support* of a signal describes where it is nonzero in time.

- **Causal signals**: zero for all negative time
- **Anti-causal signals**: zero for all positive time
- **Two-sided signals (non-causal)**: nonzero on both sides of the time origin

Causality is particularly important in real-time systems, where future values of a signal cannot be used.

Examples.

$$e^{-t}u(t) \quad \text{(causal)}, \qquad e^{t}u(-t) \quad \text{(anti-causal)}.$$

with $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ being the unit step function.

1.6 Even and Odd Signals

Signals can exhibit symmetry with respect to time reversal.

- **Even signals**: symmetric, $x(t) = x(-t)$
- **Odd signals**: antisymmetric, $x(t) = -x(-t)$

Any signal can be uniquely decomposed into even and odd components:

$$\begin{aligned} x_e(t) &= \frac{1}{2}\big(x(t) + x(-t)\big), \\ x_o(t) &= \frac{1}{2}\big(x(t) - x(-t)\big), \\ x(t) &= x_e(t) + x_o(t). \end{aligned}$$

This decomposition is especially useful in Fourier analysis and system characterization.

1.7 Deterministic and Stochastic Signals

Finally, signals can be classified based on how they are described.

Deterministic signals. A deterministic signal is completely specified by a known rule or formula. Given the same input, the signal always takes the same value.

Stochastic signals. A stochastic (random) signal cannot be predicted exactly and must be described probabilistically, using quantities such as mean, variance, autocorrelation, and spectral density.

Example. A common measurement model is

$$y[n] = x[n] + w[n],$$

where $x[n]$ is a deterministic signal and $w[n]$ represents random noise.

1.8 Energy and Power Signals

In many signal processing problems, we care about the "size" of a signal. Two closely related notions are **energy** and **average power**. These notions lead to an important taxonomy: *energy signals* versus *power signals.*

Energy

For a **continuous-time** signal $x(t)$, the energy is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

For a **discrete-time** signal $x[n]$, the energy is

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

Interpretation. Energy measures the total accumulated "signal strength" over all time. Signals that exist only for a finite time interval (or decay sufficiently fast) often have finite energy.

Example (finite-duration pulse). Consider the pulse

$$x(t) = \begin{cases} A, & 0 \le t \le T_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{T_0} A^2 dt = A^2 T_0 < \infty.$$

So this is an *energy-type* signal.

Average Power

For a **continuous-time** signal $x(t)$, the average power is defined as

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt,$$

and for a **discrete-time** signal $x[n]$,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2.$$

Interpretation. Power measures the *time-averaged* signal strength. This is appropriate for signals that persist indefinitely (e.g. periodic signals). Such signals typically have infinite energy (because the integral/sum over infinite time does not converge), but they can still have finite average power.

Example (sinusoid). Let

$$x(t) = A\cos(2\pi f_0 t).$$

Its energy is infinite (it never "dies out"), but its average power is finite:

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\cos^2(2\pi f_0 t)\,dt = A^2 \cdot \underbrace{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(2\pi f_0 t)\,dt}_{=1/2} = \frac{A^2}{2}.$$

Magnitude of a Signal

The magnitude operator $|\cdot|$ is used in the definitions of energy and power to ensure that these quantities are always nonnegative and physically meaningful.

Real-valued signals. If the signal $x(t)$ (or $x[n]$) is real-valued, then the magnitude is simply the absolute value:

$$|x(t)| = \begin{cases} x(t), & x(t) \ge 0, \\ -x(t), & x(t) < 0, \end{cases} \qquad |x(t)|^2 = x^2(t).$$

Complex-valued signals. If the signal $x(t)$ is complex-valued, it can be written as

$$x(t) = a(t) + jb(t),$$

where $a(t) = \mathrm{Re}\{x(t)\}$ is the real part and $b(t) = \mathrm{Im}\{x(t)\}$ is the imaginary part (j is the imaginary unit). The magnitude is defined as

$$|x(t)| = \sqrt{a^2(t) + b^2(t)} = \sqrt{x(t)\,x^*(t)},$$

where $x^*(t) = a(t) - jb(t)$ denotes the complex conjugate of $x(t)$. Consequently,

$$|x(t)|^2 = x(t)\,x^*(t).$$

Geometric interpretation. For complex-valued signals, $x(t)$ can be viewed as a point in the complex plane. The magnitude $|x(t)|$ corresponds to the Euclidean distance of this point from the origin.

Why magnitude squared? Using $|x(t)|^2$ in energy and power computations:

- ensures nonnegative values,
- treats positive and negative signal values symmetrically,
- extends naturally from real to complex signals,
- corresponds to physical power in many systems (e.g. electrical signals).

Energy Signals vs Power Signals

Using the above definitions, we classify signals as follows:

- **Energy signal:** $0 < E < \infty$, and $P = 0$.
- **Power signal:** $0 < P < \infty$, and $E = \infty$.

Rule of thumb.

- Signals that are *finite in duration* (or decay fast enough) are typically **energy signals**.
- Signals that are *periodic* (or persist forever with bounded amplitude) are typically **power signals**.

Signals that are neither energy nor power signals. Not all signals fall into the categories of energy signals or power signals. Some signals have:

- infinite energy, and
- infinite or undefined average power.

Such signals are classified as **neither energy nor power signals**.

Example 1: Linear signal. Consider the continuous-time linear signal

$$x(t) = t.$$

Its energy is

$$E = \int_{-\infty}^{\infty} t^2 \, dt = \infty,$$

and its average power is

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} t^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \cdot \frac{2T^3}{3} = \infty.$$

Thus, the linear signal is neither an energy signal nor a power signal.

Example 2: Exponentially growing signal. Consider

$$x(t) = e^t.$$

Both the energy and the average power diverge, so this signal is also neither an energy signal nor a power signal.

Summary.

- Energy signals: $0 < E < \infty$, $P = 0$
- Power signals: $0 < P < \infty$, $E = \infty$
- Neither: $E = \infty$ and $P = \infty$ (or undefined)

Important remark. The energy/power classification is useful but not exhaustive. Many practically relevant signals fall outside these categories, motivating alternative characterizations (e.g. RMS values, windowed energy, or statistical descriptions).

2 Common Signals and Their Properties

This section collects a set of *standard signals* that appear repeatedly in signal processing and systems theory. They serve as building blocks: complicated signals are often expressed as combinations (sums, shifts, scalings) of these elementary forms. We present both continuous-time (CT) and discrete-time (DT) versions, and emphasize the properties that are most useful later (e.g. for convolution, sampling, and system responses).

2.1 Continuous-Time (CT) Common Signals

Unit Step $u(t)$

The **unit step** models an "on" action at time $t = 0$ (Fig. 1):

$$u(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

It is used to describe *signals that start at a specific time.* For example, a signal that equals $x(t)$ only after time t_0 can be written as $x(t)\, u(t - t_0)$.

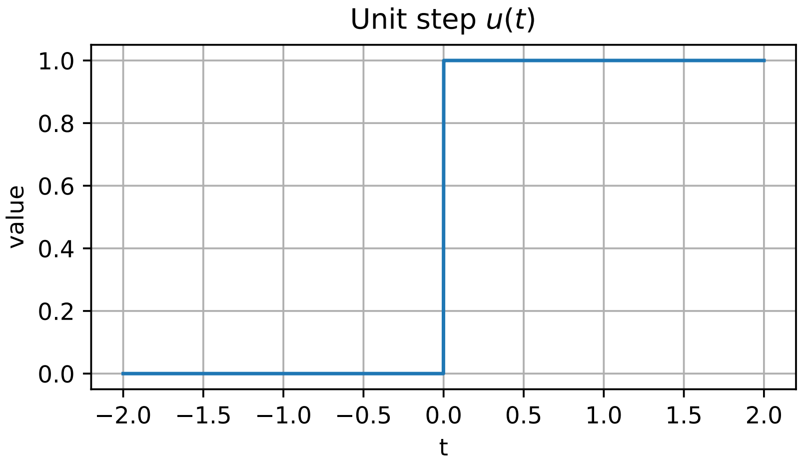

Figure 1: Unit step function $u(t)$.

Examples.

- Turning on a DC voltage source at $t = 0$: $v(t) = V_0\, u(t)$.
- A sensor that begins recording at $t = 3$: $y(t) = s(t)\, u(t - 3)$.

Time shift.

$$u(t - t_0) = \begin{cases} 1, & t \geq t_0, \\ 0, & t < t_0. \end{cases}$$

This represents a step occurring at $t = t_0$.

Rectangular Pulse

A basic finite-duration signal is the **rectangular pulse** (Fig. 2):

$$x(t) = \begin{cases} 1, & |t| \le \frac{T}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

It models "a constant value for a short time interval".

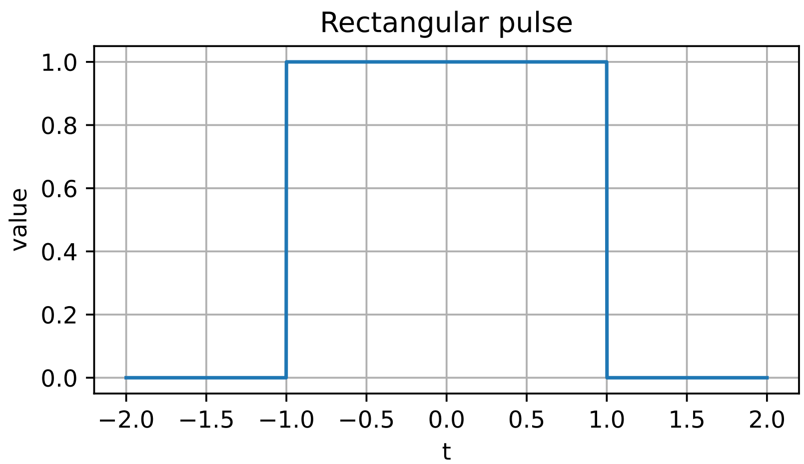

Figure 2: Rectangular pulse.

Writing pulses with steps. The same pulse can be written compactly using shifted steps:

$$x(t) = u\big(t + \tfrac{T}{2}\big) - u\big(t - \tfrac{T}{2}\big)\,.$$

This form is extremely useful when manipulating signals algebraically and when doing integrals.

Examples.

- A camera shutter that is open only for T seconds.
- A digital communication symbol held constant for one symbol period.

Ramp $r(t)$

The **ramp** is defined as (Fig. 3):

$$r(t) = t\,u(t).$$

So $r(t) = 0$ for $t < 0$ and $r(t) = t$ for $t \ge 0$. It models quantities that start at zero and then increase linearly after some time.

Examples.

- Position under constant velocity that begins at $t = 0$.
- A reference command that gradually increases instead of stepping abruptly.

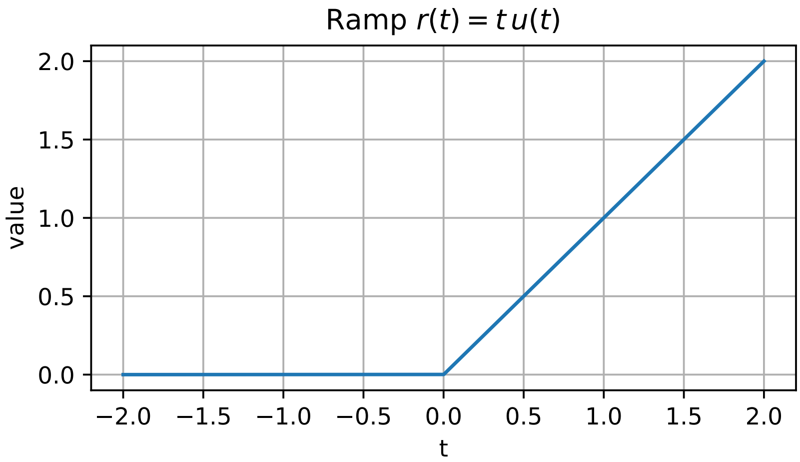

Figure 3: Ramp function $r(t)$.

Sign Function $\operatorname{sgn}(t)$

The **sign function** indicates the sign of a real-valued quantity (Fig. 4):

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

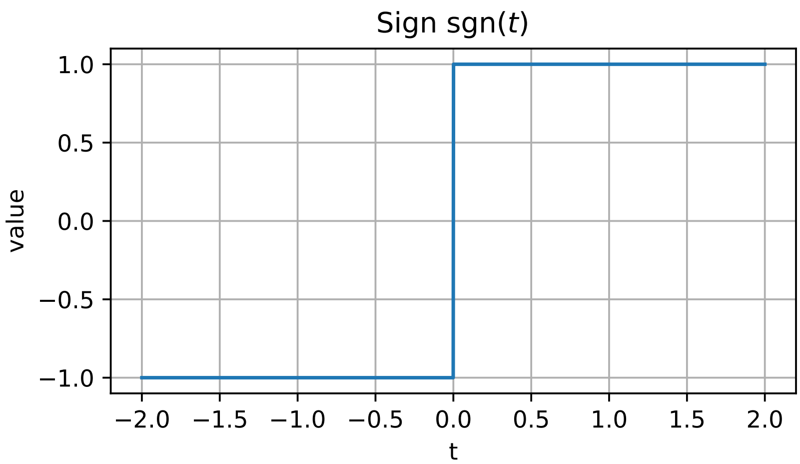

Figure 4: Sign function $\operatorname{sgn}(t)$.

Examples.

- Modeling a controller that applies $+F$ for positive error and $-F$ for negative error (bang-bang control).
- Representing a nonlinearity such as $y(t) = \operatorname{sgn}(x(t))$.

Triangular Pulse

The **triangular pulse** is a piecewise-linear signal that peaks at $t = 0$ (Fig. 5):

$$x(t) = \begin{cases} 1 - \dfrac{2|t|}{T}, & |t| \le \frac{T}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

It can be seen as a "smoothed" version of the rectangular pulse.

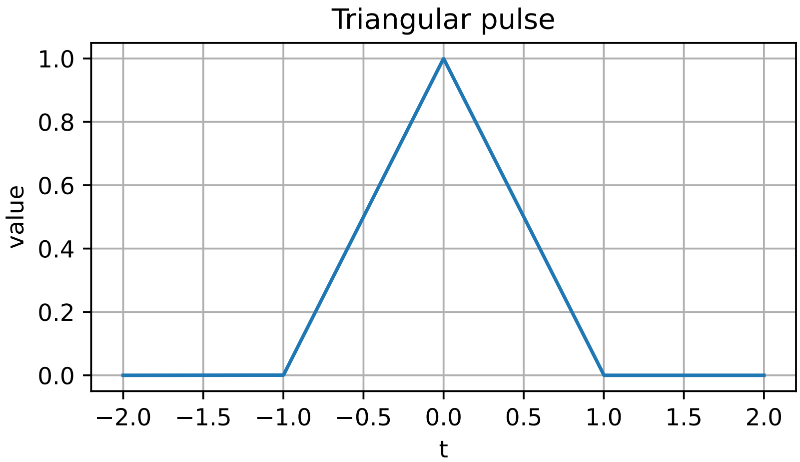

Figure 5: Triangular pulse.

Dirac Impulse $\delta(t)$

The **Dirac impulse** is not a regular function in the usual sense; it is an idealized object (a distribution) used to model instantaneous events. It is characterized by the **unit area property** (Fig. 6):

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1,$$

and by the idea that it is "concentrated" at $t = 0$.

Physical intuition. The impulse models an action that is very large in amplitude but very short in duration, such that the *total area* stays equal to 1. In physics and engineering, impulses represent idealized hits, shocks, or extremely short input bursts.

Impulse–Step Connection

The unit step and impulse are fundamentally connected:

$$\delta(t) = \frac{d}{dt} u(t), \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau.$$

Interpretation.

- The step is the *accumulation* of an instantaneous change.
- The impulse represents an *instantaneous jump* in accumulated quantity.

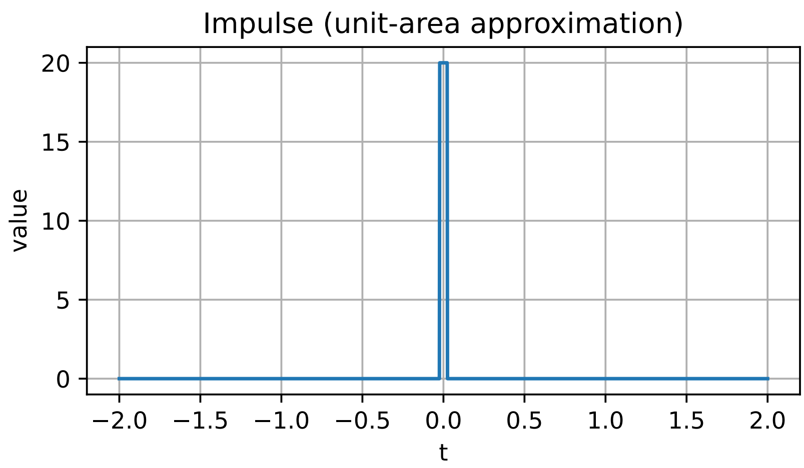

Figure 6: Dirac impulse $\delta(t)$.

Shifting and Scaling of the Impulse

Time shift.

$$\delta(t - t_0)$$

represents an impulse occurring at time $t = t_0$.

Time scaling. For any nonzero real a,

$$\delta(at) = \frac{1}{|a|}\,\delta(t),$$

and more generally

$$\delta(a(t - t_0)) = \frac{1}{|a|}\,\delta(t - t_0).$$

Key idea: time scaling changes the "height" (in an idealized sense) but preserves *unit area.*

Optional – Why $\delta(at) = \frac{1}{|a|}\delta(t)$

We justify the scaling property of the Dirac impulse by checking how it behaves inside an integral.

Step 1: Start with the integral we want to evaluate.

$$\int_{-\infty}^{\infty} \delta(at)\,dt.$$

Step 2: Apply a change of variables. Let $\tau = at$. Then $t = \tau/a$ and

$$dt = \frac{1}{|a|}\,d\tau$$

(the absolute value appears because the substitution flips integration limits when $a < 0$).

Step 3: Substitute into the integral.

$$\int_{-\infty}^{\infty} \delta(at)\,dt = \int_{-\infty}^{\infty} \delta(\tau)\,\frac{1}{|a|}\,d\tau = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\tau)\,d\tau = \frac{1}{|a|}.$$

Step 4: Enforce unit area. The defining "unit area" property of the impulse is

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

From Step 3 we found that the *unscaled* $\delta(at)$ has area

$$\int_{-\infty}^{\infty} \delta(at)\,dt = \frac{1}{|a|} \neq 1.$$

So we search for a constant gain K such that $K\,\delta(at)$ has unit area:

$$\int_{-\infty}^{\infty} K\,\delta(at)\,dt = 1.$$

Because K is a constant, it can be pulled out of the integral:

$$K\int_{-\infty}^{\infty} \delta(at)\,dt = 1.$$

Now substitute the value from Step 3:

$$K \cdot \frac{1}{|a|} = 1.$$

Solve for K:

$$K = |a|.$$

Therefore, the unit-area impulse under time scaling is

$$|a|\,\delta(at).$$

Conclusion (scaling property). The only impulse-like object with unit area after time scaling is

$$\boxed{\delta(at) = \frac{1}{|a|}\,\delta(t).}$$

Key Properties of the Impulse

Unit area.

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

Sampling (shifting) property. For a sufficiently well-behaved signal $x(t)$,

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0).$$

Multiplication by a function.

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0).$$

This expresses the fact that the impulse "picks out" the value at t_0.

Optional – Sampling and Multiplication Properties of the Impulse

The following two properties of the Dirac impulse are closely related and can be understood using the same underlying idea: the impulse "selects" the value of a signal at a specific time.

1. Sampling (shifting) property.
We start from the integral

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt.$$

The impulse $\delta(t-t_0)$ is zero everywhere except at $t = t_0$, so only the value of $x(t)$ at t_0 contributes to the integral. Intuitively, the integral "samples" the signal at $t = t_0$.
Formally, since $x(t)$ varies slowly compared to the impulse, it can be treated as constant within the infinitesimal support of $\delta(t-t_0)$:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt.$$

Using the unit-area property of the impulse,

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1,$$

we obtain

$$\boxed{\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0).}$$

2. Multiplication by a function.
Consider the product $x(t)\,\delta(t-t_0)$. Since the impulse is nonzero only at $t = t_0$, the function $x(t)$ can be replaced by its value at t_0:

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0).$$

Connection between the two properties.
The multiplication property is essentially a "pointwise" version of the sampling property. Indeed, integrating both sides of

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

over time immediately yields the sampling property:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = x(t_0).$$

2.2 Discrete-Time (DT) Common Signals

Discrete-time signals are defined only at integer indices $n \in \mathbb{Z}$. Many CT signals have direct DT analogues, but some operations change: for instance, derivatives become differences and integrals become sums.

Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

It models a sequence that turns on at index $n = 0$ (Fig. 7). A delayed turn-on is $u[n-n_0]$.

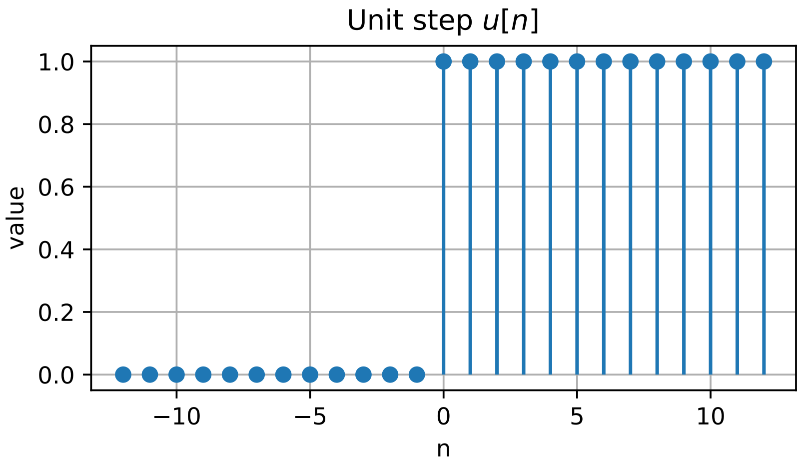

Figure 7: Unit step function $u[n]$.

Example. A sequence that starts at $n = 10$ is $x[n]u[n - 10]$.

Rectangular Pulse

$$x[n] = \begin{cases} 1, & n_1 \leq n \leq n_2, \\ 0, & \text{otherwise}. \end{cases}$$

This models a finite block of constant samples. It is often used to describe finite-length sequences (e.g. finite observation windows, see Fig. 8).

Ramp $r[n]$

$$r[n] = n\, u[n].$$

This is the discrete-time ramp: $r[n] = 0$ for $n < 0$ and $r[n] = n$ for $n \geq 0$ (Fig. 9).

Sign Function $\mathrm{sgn}[n]$

$$\mathrm{sgn}[n] = \begin{cases} 1, & n > 0, \\ 0, & n = 0, \\ -1, & n < 0. \end{cases}$$

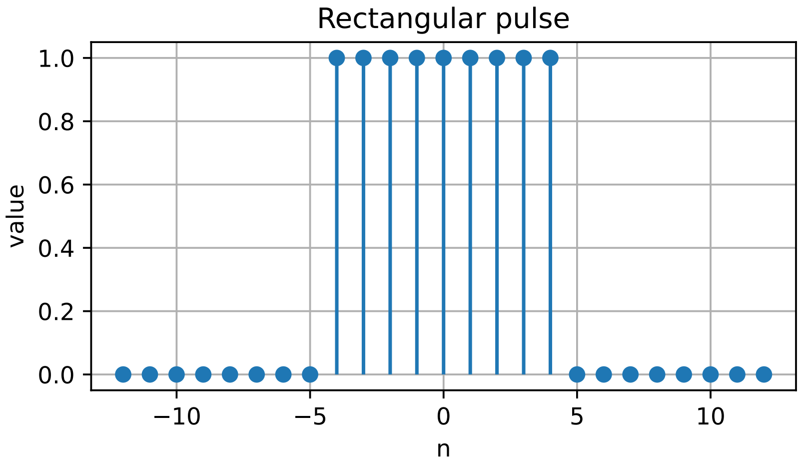

Figure 8: Rectangular pulse $x[n]$.

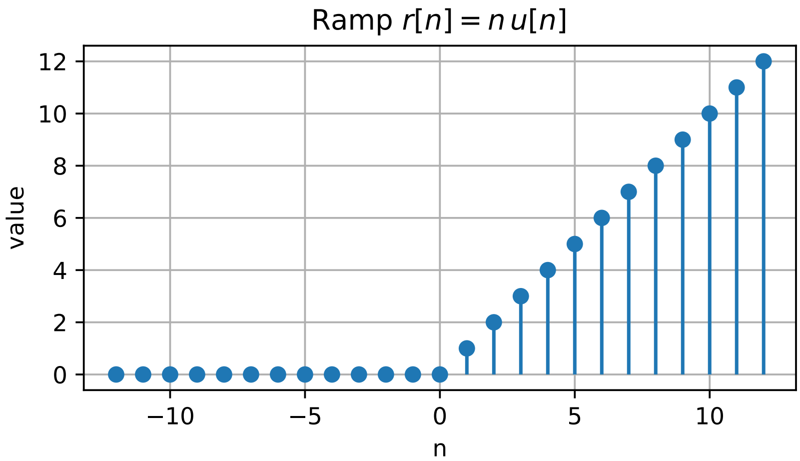

Figure 9: Ramp function $r[n]$.

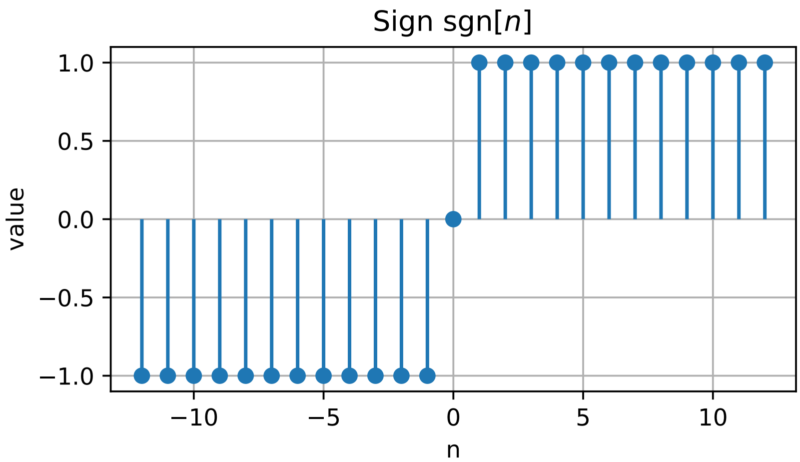

Figure 10: Sign function $\mathrm{sgn}[n]$.

Triangular Pulse

$$x[n] = \begin{cases} 1 - \dfrac{|n|}{N}, & |n| \le N, \\ 0, & \text{otherwise.} \end{cases}$$

This is a finite-length sequence that rises linearly to a peak and then decreases linearly (Fig. 11).

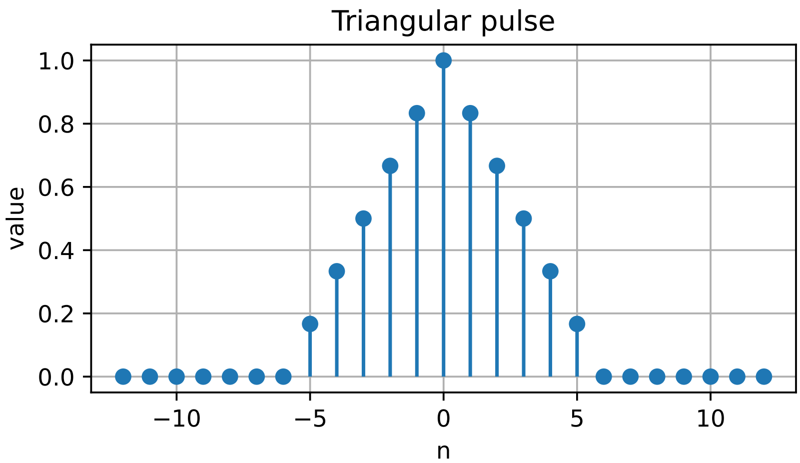

Figure 11: Triangular pulse $x[n]$.

Kronecker Impulse $\delta[n]$

The DT impulse is an *ordinary* sequence (Fig. 12):

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

It plays a central role in representing sequences as sums of shifted impulses.

Discrete-Time Impulse–Step Connection

In DT, the step is the cumulative sum of impulses:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k].$$

The shifting (sampling) property becomes:

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n - n_0] = x[n_0].$$

2.3 Complex Exponentials

Complex exponentials are among the most important signals in all of signal processing because they are eigenfunctions of linear time-invariant (LTI) systems and form the building blocks of Fourier analysis.

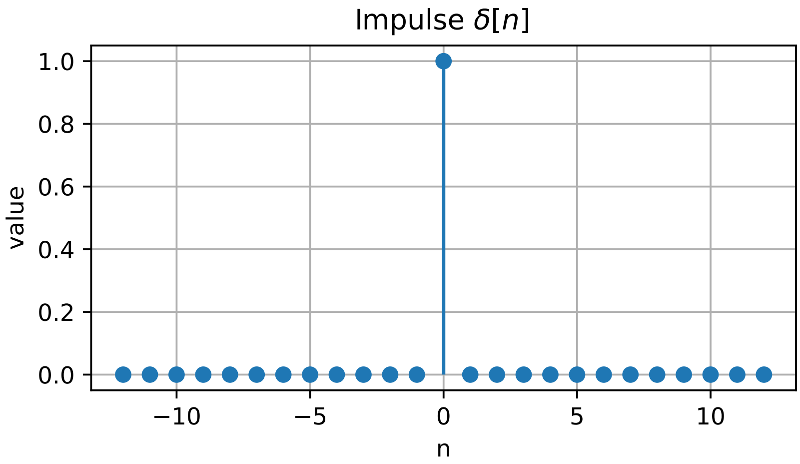

Figure 12: Kronecker impulse $\delta[n]$.

Continuous-Time Complex Exponential

$$x(t) = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} \overset{\text{Euler's formula}}{=} e^{\sigma t}\big(\cos(\omega t) + j\sin(\omega t)\big),$$

where we used the Euler's formula.

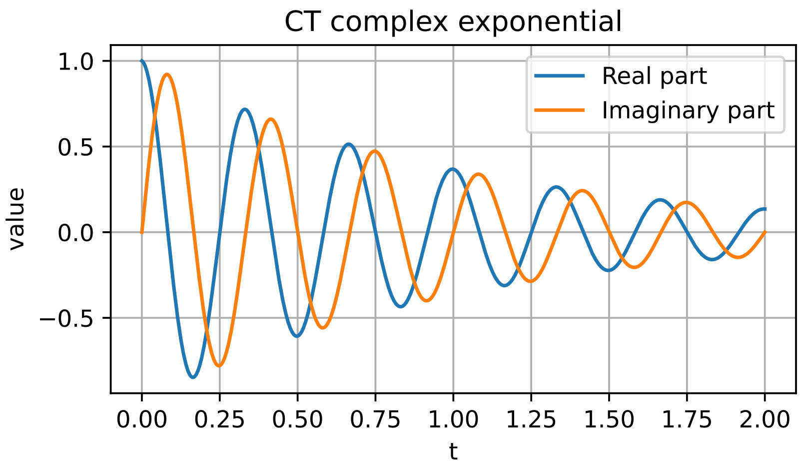

Figure 13: Continuous-time complex exponential $x(t)$.

Euler's Formula

Euler's formula connects exponentials to sinusoids:

$$e^{jx} = \cos(x) + j\sin(x).$$

It implies, for example,

$$\cos(x) = \mathrm{Re}\{e^{jx}\}, \qquad \sin(x) = \mathrm{Im}\{e^{jx}\}, \qquad \cos(x) = \frac{e^{jx} + e^{-jx}}{2}.$$

Interpretation.

- σ controls exponential growth/decay:
  - $\sigma < 0$: decaying oscillation
  - $\sigma = 0$: pure oscillation (constant magnitude)
  - $\sigma > 0$: growing oscillation
- ω is the angular frequency (rad/s), controlling how fast the signal oscillates.

Examples.

- $e^{-t}\cos(10t)$ is a damped oscillation (e.g. a stable mechanical vibration).
- $\cos(2\pi f_0 t)$ is obtained as the real part of $e^{j2\pi f_0 t}$.

Discrete-Time Complex Exponential

$$x[n] = e^{(\sigma+j\Omega)n} = e^{\sigma n}e^{j\Omega n} \overset{\text{Euler's formula}}{=} e^{\sigma n}\big(\cos(\Omega n) + j\sin(\Omega n)\big).$$

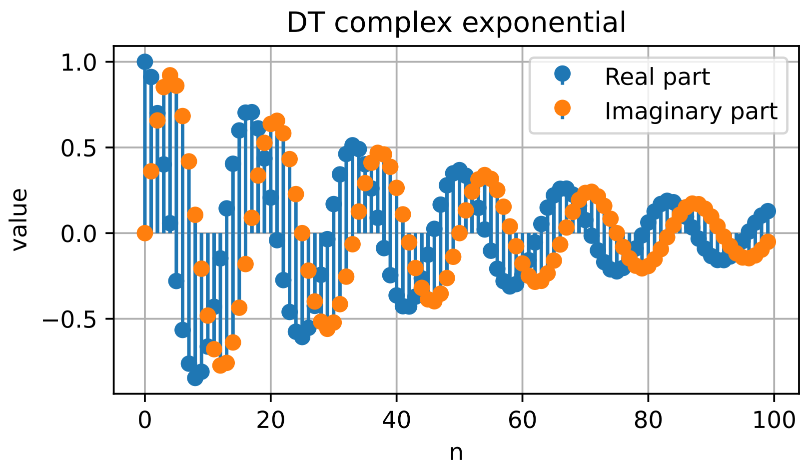

Figure 14: Discrete-time complex exponential $x[n]$.

Interpretation.

- σ controls exponential growth/decay per sample.
- Ω is the discrete-time angular frequency (rad/sample).

Periodicity note (DT). A key difference from continuous time is that DT complex exponentials can be periodic only for specific frequencies. In particular, $e^{j\Omega n}$ is periodic if and only if there exists an integer $N > 0$ such that

$$e^{j\Omega(n+N)} = e^{j\Omega n} \iff e^{j\Omega N} = 1 \iff \Omega N = 2\pi k$$

for some integer k. Equivalently, $\Omega/(2\pi)$ must be a rational number.

Why this matters. Many computations become simpler using complex exponentials, even when the final signal of interest is real-valued. This is a core reason Fourier analysis and LTI system theory are built around $e^{j\omega t}$ and $e^{j\Omega n}$.

3 Signal Transformations

In practice we rarely work with "raw" signals exactly as they are first defined. Instead, we constantly generate new signals from existing ones through *transformations*. These transformations are not just cosmetic: they are the language of system analysis (especially for LTI systems), and they also explain how signals behave under Fourier transforms (e.g. time shift $\leftrightarrow$ phase shift, time scaling $\leftrightarrow$ frequency scaling).

Throughout this section, we assume a given signal $x(\cdot)$ and we build a new signal $y(\cdot)$ by applying a transformation. We treat continuous-time (CT) and discrete-time (DT) cases side-by-side, because the *idea* is the same, but some operations (especially scaling) behave differently.

3.1 Time Shift (Delay and Advance)

Definition

Continuous time. A time shift by t_0 is defined as

$$y(t) = x(t - t_0).$$

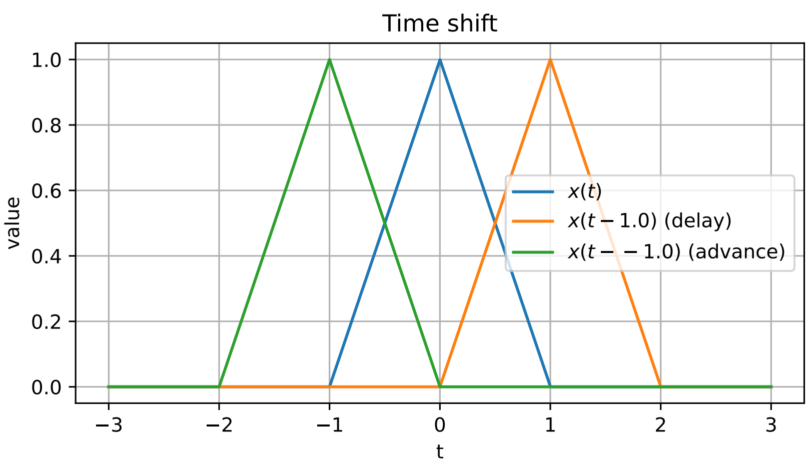

Figure 15: Time shift (delay/advance) in continuous time.

Discrete time. A shift by an integer n_0 is defined as

$$y[n] = x[n - n_0].$$

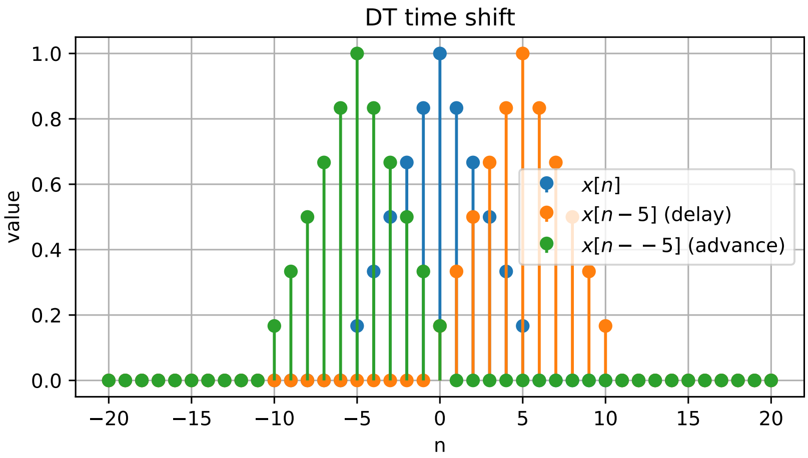

Figure 16: Time shift (delay/advance) in discrete time.

Interpretation

The key to avoid confusion is to read the argument *literally*: the output at time t is the input evaluated at $t - t_0$.

- If $t_0 > 0$, then $y(t)$ reproduces values of $x(\cdot)$ from *earlier* times. This means the whole waveform appears **later** in time: a **delay**.
- If $t_0 < 0$, the waveform appears **earlier** in time: an **advance**.

Examples.

- If $x(t)$ is a pulse centered at $t = 0$, then $x(t-3)$ is the same pulse centered at $t = 3$.
- If $x[n]$ is a sequence that "starts" at $n = 0$, then $x[n-5]$ is the same sequence starting at $n = 5$.

A common pitfall

Many initially think "$t - t_0$ means shift left." The safe rule is:

$$x(t - t_0) \text{ shifts the signal to the right by } t_0.$$

A quick check: where does the event that used to be at $t = 0$ move? Solve $t - t_0 = 0 \Rightarrow t = t_0$. So it moves to t_0.

3.2 Time Reversal (Folding)

Definition

Continuous time.

$$y(t) = x(-t).$$

Discrete time.

$$y[n] = x[-n].$$

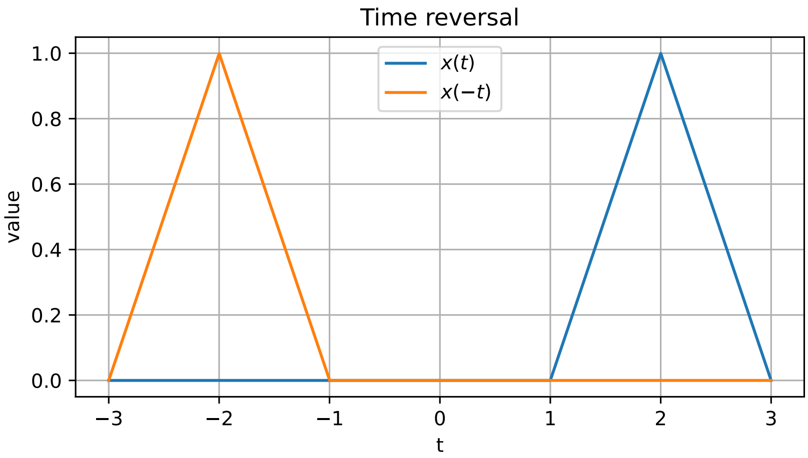

Figure 17: Time reversal in continuous time.

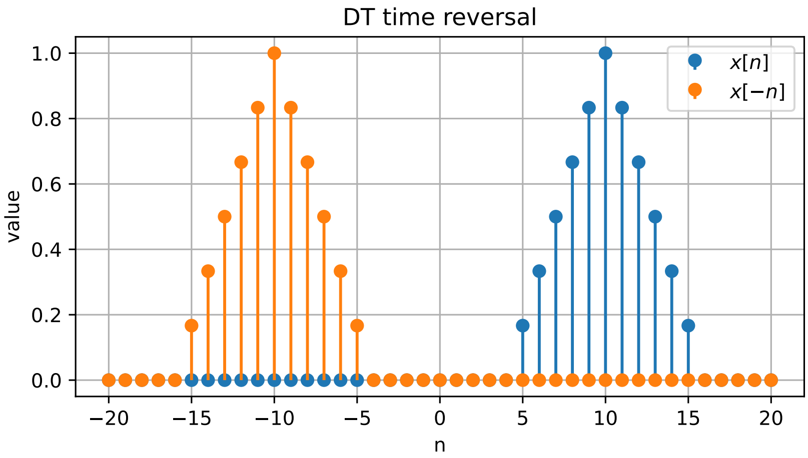

Figure 18: Time reversal in discrete time.

Interpretation

Time reversal mirrors the signal around the origin:

- what was at $t = +1$ moves to $t = -1$,
- what was at $t = +2$ moves to $t = -2$, etc.

Graphically, you "flip" the signal around the vertical axis at $t = 0$ (or $n = 0$).

Examples.

- If $x(t)$ is causal (zero for $t < 0$), then $x(-t)$ is anti-causal (zero for $t > 0$).
- If $x(t)$ is even, then $x(-t) = x(t)$ (nothing changes).
- If $x(t)$ is odd, then $x(-t) = -x(t)$ (it flips sign).

Shift and reversal combined

A very common pattern is

$$y(t) = x(-(t - t_0)) = x(t_0 - t).$$

This corresponds to reversing around $t = 0$ and then shifting (or equivalently reversing around the point $t = t_0/2$). When sketching such signals, it is usually easiest to do it in two steps: reversal first, shift second.

3.3 Time Scaling

Definition (Continuous-Time)

For continuous-time signals, time scaling is defined by

$$y(t) = x(at),$$

where $a \neq 0$.

Interpretation

Time scaling changes *how fast* the signal evolves:

- $|a| > 1$: **time compression**. The signal changes faster and appears "squeezed" toward $t = 0$.
- $0 < |a| < 1$: **time expansion**. The signal changes slower and appears "stretched".
- $a < 0$: scaling *plus* reversal (because $t \mapsto at$ flips the time axis when $a < 0$).

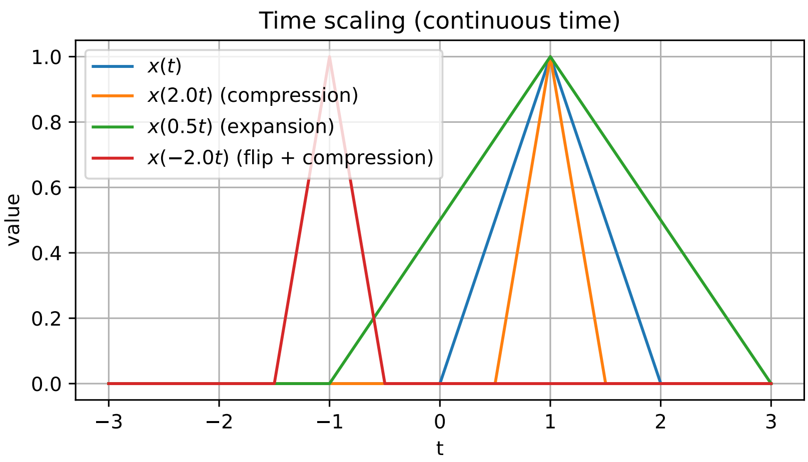

Figure 19: Time scaling in continuous time.

Examples.

- If $x(t) = \cos(2\pi f_0 t)$, then $x(2t) = \cos(2\pi(2f_0)t)$: frequency doubles (faster oscillations).
- If $x(t)$ is a pulse of width T, then $x(2t)$ has width $T/2$ (compressed), while $x(0.5t)$ has width $2T$ (expanded).

Why scaling compresses or expands

Suppose a distinctive feature of $x(t)$ occurs at time $t = t_1$ (for example, a peak). In the scaled signal $y(t) = x(at)$, the same feature occurs when

$$at = t_1 \quad \Rightarrow \quad t = \frac{t_1}{a}.$$

So feature times are divided by a:

- if $a = 2$, features move to half the time (closer to zero) $\Rightarrow$ compression,
- if $a = 0.5$, features move to twice the time $\Rightarrow$ expansion.

Discrete-time scaling

In discrete time, n must remain an integer. The expression $x[an]$ only makes sense when an is an integer index:

- If a is an integer (e.g. $a = 2$), then $x[2n]$ is well-defined but it *selects* every second sample (downsampling).
- If $0 < a < 1$ (e.g. $a = 1/2$), then $x[n/2]$ is not defined for odd n unless we introduce interpolation, which is a separate topic.

For this reason, "time scaling" in discrete time is typically discussed in terms of *downsampling* and *upsampling* (and interpolation), rather than the simple $x(at)$ formula.

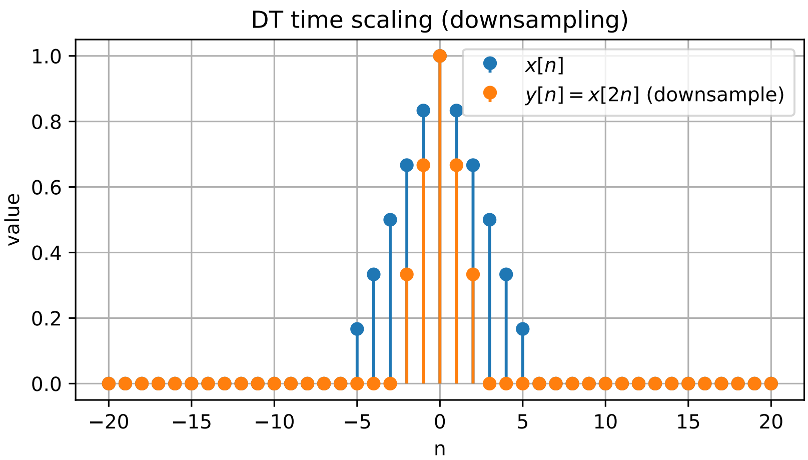

Figure 20: Time scaling in discrete time (downsampling).

3.4 Amplitude Scaling (Vertical Scaling)

Definition

Amplitude scaling multiplies the signal values by a constant α.

Continuous time.

$$y(t) = \alpha\, x(t).$$

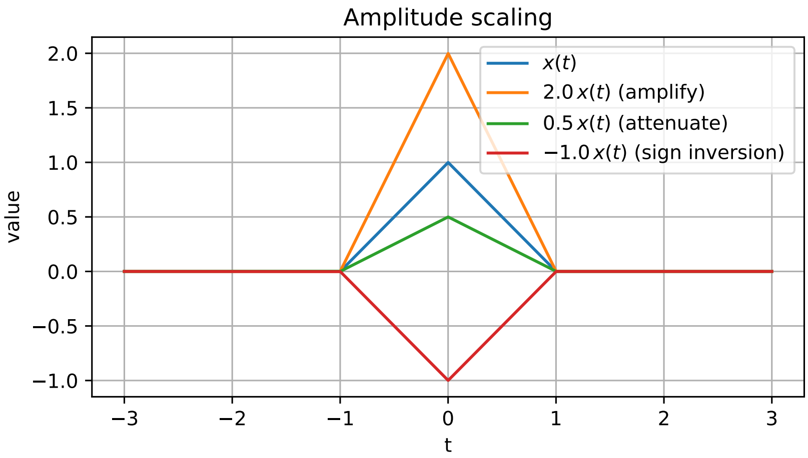

Figure 21: Amplitude scaling in continuous time.

Discrete time.

$$y[n] = \alpha\, x[n].$$

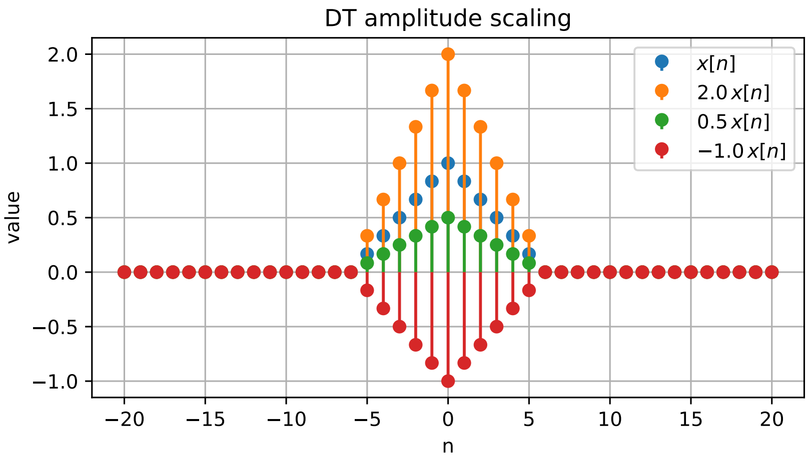

Figure 22: Amplitude scaling in discrete time.

Interpretation

Amplitude scaling does not change *when* events occur, only *how large* they are.

- $|\alpha| > 1$: amplification
- $0 < |\alpha| < 1$: attenuation
- $\alpha < 0$: sign inversion (vertical flip)

Examples.

- Audio volume knob: approximately $y(t) = \alpha x(t)$.
- Sensor calibration: if a sensor reports $y(t) = 2x(t)$, we can correct by scaling with $\alpha = \frac{1}{2}$.

Effect on energy and power

Because energy and power use $|x(\cdot)|^2$, scaling a signal by α scales these measures by $|\alpha|^2$:

$$E_y = \int_{-\infty}^{\infty} |\alpha x(t)|^2 dt = |\alpha|^2 \int_{-\infty}^{\infty} |x(t)|^2 dt = |\alpha|^2 E_x,$$

and similarly (when defined) $P_y = |\alpha|^2 P_x$.

3.5 Putting Transformations Together

In real problems, we often combine transformations. A general and extremely common form is

$$y(t) = \alpha\, x\big(a(t - t_0)\big),$$

which applies (in order):

- shift by t_0,
- scale by a (and possibly reverse if $a < 0$),
- amplitude-scale by α.

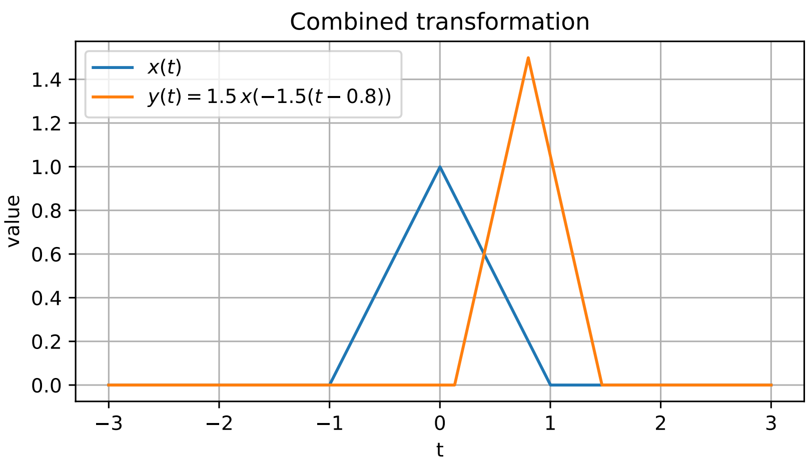

Figure 23: Combined transformation in continuous time.

Recommended sketching procedure. When asked to sketch $y(t) = \alpha\, x(a(t - t_0))$, avoid trying to do everything at once. Do it in steps:

1. Start from the graph of $x(t)$.
2. Apply the shift to get $x(t - t_0)$.
3. Apply time scaling to get $x(a(t - t_0))$.
4. Apply amplitude scaling to get $\alpha x(a(t - t_0))$.

This systematic approach prevents sign mistakes and confusion about left/right shifts.

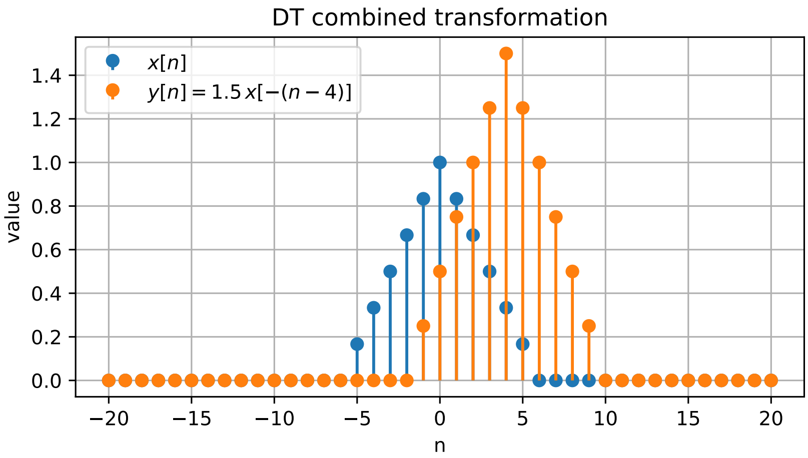

Figure 24: Combined transformation in discrete time.

4 Energy and Power

Energy and power quantify the "size" of a signal, but they do so in different ways.

- **Energy** measures the *total accumulated* squared magnitude of the signal over all time.
- **Average power** measures the *time-averaged* squared magnitude.

These definitions are fundamental because many practical questions reduce to "how big is the signal?" and because they connect directly to physical power in electrical/mechanical systems and to RMS values.

4.1 Definitions (CT and DT)

Continuous-time (CT)

For a continuous-time signal $x(t)$, we define energy (E_x) and average power (P_x) as:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)x^*(t) \, dt,$$
$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x^*(t) \, dt,$$

provided the limit for P_x exists. x^* denotes the complex conjugate of x.

Interpretation.

- E_x is a (possibly infinite) *total* quantity.
- P_x is an *average per unit time* (a long-term average).

Discrete-time (DT)

For a discrete-time signal $x[n]$, we define energy (E_x) and average power (P_x) as:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n],$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]x^*[n],$$

provided the limit exists.

4.2 Energy Signals vs Power Signals

The classical classification is:

- **Energy signal:** $0 < E_x < \infty$, and $P_x = 0$.
- **Power signal:** $0 < P_x < \infty$, and $E_x = \infty$.
- **Neither:** energy diverges and power is infinite or undefined.

Quick examples.

- Finite pulse $\Rightarrow$ energy signal.
- Sinusoid $\Rightarrow$ power signal.
- Ramp $x(t) = t \Rightarrow$ neither (both diverge).

4.3 How Energy and Power Behave Under Transformations

A major reason we study transformations is that we often know E_x or P_x for a basic signal and want to deduce E_y or P_y for a transformed one.

4.3.1 Time Shift: Energy and Power Are Unchanged

Continuous time. Let

$$y(t) = x(t - t_0).$$

Energy.

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t-t_0)|^2\, dt.$$

Let $\tau = t - t_0$, so $dt = d\tau$. The integration limits remain $-\infty$ to ∞, hence

$$E_y = \int_{-\infty}^{\infty} |x(\tau)|^2\, d\tau = E_x.$$

Average power.

$$P_y = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t-t_0)|^2\, dt.$$

Using the same substitution $\tau = t - t_0$ gives

$$P_y = \lim_{T\to\infty} \frac{1}{2T} \int_{-T-t_0}^{T-t_0} |x(\tau)|^2 \, d\tau.$$

For large T, shifting the integration window by a fixed t_0 does not affect the long-term average (assuming the limit exists), therefore

$$P_y = P_x.$$

Discrete time. Let

$$y[n] = x[n - n_0].$$

Energy.

$$E_y = \sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n - n_0]|^2.$$

Let $k = n - n_0$. As n runs over all integers, so does k. Thus

$$E_y = \sum_{k=-\infty}^{\infty} |x[k]|^2 = E_x.$$

Average power.

$$P_y = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n - n_0]|^2.$$

Again, shifting the averaging window by a fixed n_0 does not change the time average (if the limit exists), so

$$P_y = P_x.$$

Key message: time shift changes *when* the signal happens, not its values; therefore it does not change energy or average power.

4.3.2 Time Scaling: Energy Changes, Power Stays the Same

Let

$$y(t) = x(at), \qquad a \neq 0.$$

Energy.

$$E_y = \int_{-\infty}^{\infty} |x(at)|^2 \, dt.$$

Let $\tau = at$, hence $dt = \frac{1}{|a|} \, d\tau$ (the absolute value ensures correctness when $a < 0$). Then

$$E_y = \int_{-\infty}^{\infty} |x(\tau)|^2 \frac{d\tau}{|a|} = \frac{1}{|a|} \int_{-\infty}^{\infty} |x(\tau)|^2 \, d\tau = \frac{1}{|a|} E_x.$$

Why this makes sense. If $|a| > 1$, the signal is compressed in time (it "happens faster"), so the total accumulated energy decreases. If $0 < |a| < 1$, the signal is stretched in time, so it lasts longer and energy increases.

Average power.

$$P_y = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(at)|^2 \, dt.$$

Let $\tau = at$:

$$P_y = \lim_{T\to\infty} \frac{1}{2T} \cdot \frac{1}{|a|} \int_{-|a|T}^{|a|T} |x(\tau)|^2 \, d\tau.$$

Now rewrite the prefactor:

$$\frac{1}{2T} \cdot \frac{1}{|a|} = \frac{1}{2(|a|T)}.$$

Therefore

$$P_y = \lim_{T\to\infty} \frac{1}{2(|a|T)} \int_{-|a|T}^{|a|T} |x(\tau)|^2 \, d\tau.$$

As $T \to \infty$, the window length $|a|T \to \infty$, so this is exactly the definition of P_x (assuming it exists). Hence

$$P_y = P_x.$$

Key message: time scaling redistributes the same values over time, changing total energy, but not the long-term average power.

4.3.3 Discrete-time Downsampling (Decimation)

Let

$$y[n] = x[Mn], \qquad M \in \mathbb{Z},\ M > 1.$$

Energy.

$$E_y = \sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |x[Mn]|^2.$$

This sum includes only a subset of the terms of

$$E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2,$$

namely those indices k that are multiples of M:

$$E_y = \sum_{k\in\{\dots,-2M,-M,0,M,2M,\dots\}} |x[k]|^2 \ \le\ \sum_{k=-\infty}^{\infty} |x[k]|^2 = E_x.$$

So downsampling cannot increase energy; it usually reduces it by "discarding" samples.

Power (intuition). For power, we average over time indices. Downsampling changes what "one unit of time" means in the new sequence, and it also changes which samples are present. In general, power can change.

Important special case (periodic / power signals). If $x[n]$ is periodic and M preserves periodicity in a compatible way, the average power can remain unchanged.

4.3.4 Amplitude Scaling: Energy and Power Scale by $|\alpha|^2$

Continuous time. Let

$$y(t) = \alpha x(t).$$

Energy.

$$E_y = \int_{-\infty}^{\infty} |\alpha x(t)|^2 \, dt = |\alpha|^2 \int_{-\infty}^{\infty} |x(t)|^2 \, dt = |\alpha|^2 E_x.$$

Average power.

$$P_y = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\alpha x(t)|^2 \, dt = |\alpha|^2 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = |\alpha|^2 P_x.$$

Discrete time. Let

$$y[n] = \alpha x[n].$$

Energy.

$$E_y = \sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |\alpha x[n]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n]|^2 = |\alpha|^2 E_x.$$

Average power.

$$P_y = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |y[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\alpha x[n]|^2.$$

Pull out the constant:

$$P_y = |\alpha|^2 \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = |\alpha|^2 P_x,$$

provided the time-average limit exists.

4.4 Root Mean Square (RMS) Value

In addition to energy and average power, an extremely important quantity in signal processing is the **root mean square (RMS)** value. RMS provides a practical measure of the "effective amplitude" of a signal and is widely used in engineering applications such as audio processing, vibration analysis, and electrical power systems.

Motivation: Why RMS?

Energy and power answer different questions:

- Energy tells us the total accumulated signal strength over all time.
- Average power tells us the long-term average signal strength.

However, in many applications we want a quantity that has the *same units as the signal itself* (volts, meters/second, etc.), not squared units.

This motivates RMS.

Definition (Continuous-Time)

For a continuous-time signal $x(t)$, the RMS value over a finite interval $[-T, T]$ is defined as

$$x_{\text{RMS}}(T) = \sqrt{\frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}.$$

The RMS value over infinite time (when it exists) is

$$x_{\text{RMS}} = \sqrt{\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}.$$

Definition (Discrete-Time)

For a discrete-time signal $x[n]$, the RMS value over the window $[-N, N]$ is

$$x_{\text{RMS}}(N) = \sqrt{\frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2}.$$

Over infinite time (when the limit exists),

$$x_{\text{RMS}} = \sqrt{\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2}.$$

Connection Between RMS and Average Power

Observe that the average power is defined as

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt.$$

Therefore, whenever P_x exists,

$$\boxed{x_{\text{RMS}} = \sqrt{P_x}.}$$

Interpretation. RMS is simply the square root of the average power, expressed in the same units as the signal.

Physical Meaning: Effective Amplitude

For electrical signals, RMS voltage is directly related to delivered power. For example, a sinusoidal voltage

$$v(t) = V_0 \cos(\omega t)$$

has RMS value

$$v_{\text{RMS}} = \frac{V_0}{\sqrt{2}}.$$

A 230 V household outlet is quoted in RMS, meaning the peak voltage is approximately

$$V_{\max} = 230\sqrt{2} \approx 325 \text{ V}.$$

Key takeaway. A sinusoid with peak amplitude A has an effective (RMS) amplitude of $A/\sqrt{2}$.

Worked Example: RMS of a Finite Pulse

Consider the rectangular pulse

$$x(t) = \begin{cases} A, & 0 \le t \le T_0, \\ 0, & \text{otherwise.} \end{cases}$$

Its RMS value over the interval $[0, T_0]$ is

$$x_{\mathrm{RMS}} = \sqrt{\frac{1}{T_0} \int_0^{T_0} A^2 \, dt} = \sqrt{\frac{1}{T_0} \cdot A^2 T_0} = A.$$

So a constant-amplitude pulse has RMS equal to its amplitude.

Why RMS Matters in Signal Processing

RMS is widely used because:

- it measures signal strength in the same units as the signal,
- it is directly linked to power,
- it provides a meaningful "average amplitude",
- it connects naturally to the decibel scale:

$$20 \log_{10} \left(\frac{x_{\mathrm{RMS},2}}{x_{\mathrm{RMS},1}} \right).$$

Summary.

$$\boxed{x_{\mathrm{RMS}} = \sqrt{\text{average of } |x|^2}} \qquad \Longleftrightarrow \qquad \boxed{x_{\mathrm{RMS}} = \sqrt{P_x}.}$$

4.5 Worked Examples

The goal of these examples is to illustrate the computation of energy and power for specific signals, as well as to demonstrate how transformations affect these quantities. Key steps include:

- setting up the correct limits/integration bounds,
- using standard integral identities,
- interpreting the result (energy vs power signal),
- checking consistency with the transformation rules.

Example 1 (CT): Energy Under Time Scaling

Let

$$x(t) = e^{-t}u(t),$$

where $u(t)$ is the unit step. Because of $u(t)$, the signal is zero for $t < 0$, hence all integrals reduce to the interval $[0, \infty)$.

Step 1: Compute $|x(t)|^2$. Here $x(t)$ is real and nonnegative for $t \geq 0$, so

$$|x(t)|^2 = x^2(t) = e^{-2t}u(t).$$

Step 2: Write the energy integral.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} e^{-2t}u(t) \, dt = \int_{0}^{\infty} e^{-2t} \, dt.$$

Step 3: Evaluate the integral. Compute the antiderivative:

$$\int e^{-2t} \, dt = -\frac{1}{2}e^{-2t} + C.$$

Apply limits:

$$E_x = \left[-\frac{1}{2}e^{-2t} \right]_0^{\infty} = \lim_{b \to \infty} \left(-\frac{1}{2}e^{-2b} \right) - \left(-\frac{1}{2}e^{0} \right).$$

Now $e^{-2b} \to 0$ as $b \to \infty$, hence

$$E_x = 0 + \frac{1}{2} = \frac{1}{2}.$$

Step 4: Time-scale the signal. Define

$$y(t) = x(2t) = e^{-2t}u(2t).$$

Since $u(2t) = u(t)$ (since $2 > 0$), we have

$$y(t) = e^{-2t}u(t).$$

Step 5: Compute $|y(t)|^2$ and its energy.

$$|y(t)|^2 = e^{-4t}u(t), \qquad E_y = \int_{-\infty}^{\infty} |y(t)|^2 \, dt = \int_{0}^{\infty} e^{-4t} \, dt.$$

Compute:

$$\int e^{-4t} \, dt = -\frac{1}{4}e^{-4t} + C,$$

so

$$E_y = \left[-\frac{1}{4}e^{-4t} \right]_0^{\infty} = \lim_{b \to \infty} \left(-\frac{1}{4}e^{-4b} \right) - \left(-\frac{1}{4}e^{0} \right) = 0 + \frac{1}{4} = \frac{1}{4}.$$

Step 6: Check against the general rule. For CT time scaling $y(t) = x(at)$, the energy scales as $E_y = \frac{1}{|a|}E_x$. Here $a = 2$, so

$$\frac{1}{|2|}E_x = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

which matches the direct computation.

Interpretation. $y(t) = x(2t)$ is a *compressed* version of $x(t)$: the signal "happens twice as fast", so the total accumulated energy is halved.

Example 2 (CT): Power of a Sinusoid

Let

$$x(t) = A\cos(\omega_0 t),$$

with $A \in \mathbb{R}$. This signal persists forever, so we expect its energy to diverge, but its average power to be finite.

Step 1: Energy diverges.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} A^2\cos^2(\omega_0 t)\,dt.$$

Since $\cos^2(\cdot) \geq 0$ and its average value is not zero, the integral grows without bound. More explicitly, for any $T > 0$,

$$E_x \geq \int_{-T}^{T} A^2\cos^2(\omega_0 t)\,dt.$$

It is easy to see that the right-hand side grows proportionally to T (hence diverges as $T \to \infty$).

Step 2: Set up average power.

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\cos^2(\omega_0 t)\,dt.$$

Factor out the constant A^2:

$$P_x = A^2 \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(\omega_0 t)\,dt.$$

Step 3: Use a trigonometric identity. Use

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}.$$

With $\theta = \omega_0 t$:

$$\cos^2(\omega_0 t) = \frac{1+\cos(2\omega_0 t)}{2}.$$

Step 4: Integrate exactly over $[-T, T]$.

$$\int_{-T}^{T} \cos^2(\omega_0 t)\,dt = \int_{-T}^{T} \frac{1+\cos(2\omega_0 t)}{2}\,dt = \frac{1}{2}\int_{-T}^{T} 1\,dt + \frac{1}{2}\int_{-T}^{T} \cos(2\omega_0 t)\,dt.$$

Compute each term:

$$\frac{1}{2}\int_{-T}^{T} 1\,dt = \frac{1}{2}(2T) = T.$$

For the cosine term,

$$\int \cos(2\omega_0 t)\,dt = \frac{1}{2\omega_0}\sin(2\omega_0 t),$$

hence

$$\frac{1}{2}\int_{-T}^{T} \cos(2\omega_0 t)\,dt = \frac{1}{2}\left[\frac{1}{2\omega_0}\sin(2\omega_0 t)\right]_{-T}^{T} = \frac{1}{4\omega_0}\Big(\sin(2\omega_0 T) - \sin(-2\omega_0 T)\Big).$$

Using $\sin(-x) = -\sin(x)$:

$$\sin(2\omega_0 T) - \sin(-2\omega_0 T) = \sin(2\omega_0 T) + \sin(2\omega_0 T) = 2\sin(2\omega_0 T),$$

so

$$\frac{1}{2}\int_{-T}^{T} \cos(2\omega_0 t)\,dt = \frac{1}{4\omega_0}\cdot 2\sin(2\omega_0 T) = \frac{1}{2\omega_0}\sin(2\omega_0 T).$$

Therefore,

$$\int_{-T}^{T} \cos^2(\omega_0 t)\,dt = T + \frac{1}{2\omega_0}\sin(2\omega_0 T).$$

Step 5: Divide by $2T$ and take the limit.

$$P_x = A^2 \lim_{T\to\infty} \frac{1}{2T}\left(T + \frac{1}{2\omega_0}\sin(2\omega_0 T)\right) = A^2 \lim_{T\to\infty}\left(\frac{1}{2} + \frac{1}{4\omega_0 T}\sin(2\omega_0 T)\right).$$

Since $|\sin(2\omega_0 T)| \le 1$,

$$\left|\frac{1}{4\omega_0 T}\sin(2\omega_0 T)\right| \le \frac{1}{4|\omega_0|T} \xrightarrow[T\to\infty]{} 0,$$

hence

$$\boxed{P_x = \frac{A^2}{2}.}$$

Interpretation. A sinusoid is a **power signal**: it has infinite energy (it never ends), but finite average power. Also note the physical connection: for a sinusoidal voltage across a $1\,\Omega$ resistor, average electrical power is exactly the average of $v^2(t)$, which is $\frac{A^2}{2}$.

Example 3 (DT): Energy Signal vs Power Signal

Part A: Finite-length sequence (energy signal). Define

$$x[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Step 1: Compute energy. By definition,

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

Since $x[n] = 0$ outside $0 \le n \le 4$, the infinite sum reduces to a finite sum:

$$E_x = \sum_{n=0}^{4} |1|^2 = \sum_{n=0}^{4} 1 = 5.$$

Thus $0 < E_x < \infty$, and $x[n]$ is an **energy signal**.

Step 2: Compute average power.

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2.$$

For large N, the sum contains exactly five ones (for $n = 0, 1, 2, 3, 4$) and zeros elsewhere:

$$\sum_{n=-N}^{N} |x[n]|^2 = 5 \quad \text{for all } N \ge 4.$$

Therefore,

$$P_x = \lim_{N\to\infty} \frac{5}{2N+1} = 0.$$

So energy signals have zero average power.

Part B: Infinite-length periodic sequence (power signal). Let

$$x[n] = \cos(\Omega_0 n).$$

Step 1: Energy diverges. Energy is

$$E_x = \sum_{n=-\infty}^{\infty} \cos^2(\Omega_0 n).$$

Since $\cos^2(\cdot) \ge 0$ and does not go to zero as $|n| \to \infty$, the sum does not converge; in fact it grows without bound as we extend the summation limits. Thus $E_x = \infty$.

Step 2: Compute average power.

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2(\Omega_0 n).$$

Use the identity

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2},$$

to get

$$\cos^2(\Omega_0 n) = \frac{1+\cos(2\Omega_0 n)}{2}.$$

Hence,

$$\sum_{n=-N}^{N} \cos^2(\Omega_0 n) = \frac{1}{2}\sum_{n=-N}^{N} 1 + \frac{1}{2}\sum_{n=-N}^{N} \cos(2\Omega_0 n).$$

The first sum is simple:

$$\frac{1}{2}\sum_{n=-N}^{N} 1 = \frac{1}{2}(2N+1).$$

So

$$\sum_{n=-N}^{N} \cos^2(\Omega_0 n) = \frac{1}{2}(2N+1) + \frac{1}{2}\sum_{n=-N}^{N} \cos(2\Omega_0 n).$$

Divide by $2N+1$:

$$\frac{1}{2N+1}\sum_{n=-N}^{N} \cos^2(\Omega_0 n) = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2N+1}\sum_{n=-N}^{N} \cos(2\Omega_0 n).$$

Step 3: Argue the cosine-average term vanishes (typical case). For most Ω_0 (i.e. when $2\Omega_0$ is not a multiple of 2π), the samples of $\cos(2\Omega_0 n)$ oscillate and the average tends to zero:

$$\lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} \cos(2\Omega_0 n) = 0.$$

Assuming this typical case, we obtain

$$\boxed{P_x = \frac{1}{2}.}$$

Special cases. If $\Omega_0 = 0$ (or $\Omega_0 = \pi$), then $\cos(\Omega_0 n)$ becomes a constant or alternating constant in magnitude, and the same conclusion $P_x = \frac{1}{2}$ still holds for $\cos^2(\Omega_0 n)$ (it is identically 1 when $\Omega_0 = 0$ and identically 1 when $\Omega_0 = \pi$ after squaring). The key point remains: periodic sequences are typically **power signals**.

Interpretation. A periodic DT sinusoid is a **power signal**: it has infinite energy (infinite duration) but finite nonzero average power.

5 The Decibel (dB) Scale

Many signal processing quantities span a *huge* range of values. For example, an audio signal may contain components whose amplitudes differ by factors of 10^3 or more, and power spectral densities may differ by factors of 10^6 or 10^{12}. Plotting such numbers on a linear scale often hides small (but important) components.

The **decibel (dB) scale** is a logarithmic scale designed to represent **ratios** compactly and to match how many physical systems (and human perception, e.g. hearing) respond to relative changes.

5.1 dB Expresses Ratios, Not Absolute Values

A common misconception is that "dB is a unit." In fact, dB is primarily a way to express a **ratio** between two quantities on a log scale.

Power ratio (definition)

If P_1 and P_2 are powers (or power-like quantities), the ratio in decibels is

$$\mathrm{dB} = 10\log_{10}\left(\frac{P_2}{P_1}\right).$$

Key properties.

- If $P_2 = P_1$, then $\mathrm{dB} = 0$.
- If $P_2 > P_1$, then $\mathrm{dB} > 0$ (gain).
- If $P_2 < P_1$, then $\mathrm{dB} < 0$ (loss).

Amplitude (magnitude) ratio

Often we compare *amplitudes* (e.g. voltages, currents, magnitudes of Fourier transforms). If an amplitude-like quantity is A and power is proportional to A^2, then:

$$\frac{P_2}{P_1} = \left(\frac{A_2}{A_1}\right)^2.$$

Plugging into the power definition:

$$\begin{aligned}\mathrm{dB} = 10\log_{10}\left(\frac{P_2}{P_1}\right) &= 10\log_{10}\left(\left(\frac{A_2}{A_1}\right)^2\right)\\ &= 10\cdot 2\log_{10}\left(\frac{A_2}{A_1}\right) = \boxed{20\log_{10}\left(\frac{A_2}{A_1}\right)}.\end{aligned}$$

Why 20 and not 10? Because power scales with the *square* of amplitude: $P \propto A^2$. The factor 20 appears when you convert an amplitude ratio into an equivalent power ratio.

Important condition (when amplitude dB is valid). The formula $20\log_{10}(A_2/A_1)$ assumes that A_1 and A_2 are measured under the *same reference conditions* so that the proportionality between P and A^2 is consistent (e.g. same impedance/load in electrical settings). When the reference differs, you must compute power explicitly.

5.2 Rules of Thumb (Build Intuition)

Because logarithms are not always intuitive at first, it is useful to memorize a few key reference points.

Power ratios

$+3$ dB is about $\times 2$ in power.

$$10\log_{10}(2) \approx 3.0103 \text{ dB}.$$

So doubling power corresponds to approximately $+3$ dB.

$+10$ dB is exactly $\times 10$ in power.

$$10\log_{10}(10) = 10 \text{ dB}.$$

Amplitude ratios

$+20$ dB is exactly $\times 10$ in amplitude.

$$20\log_{10}(10) = 20 \text{ dB}.$$

$+6$ dB is about $\times 2$ in amplitude. Because $20\log_{10}(2) \approx 6.0206$ dB, doubling amplitude corresponds to about $+6$ dB.

Negative dB values

A negative dB value simply means the ratio is less than one. For example, if $P_2 = 0.1P_1$, then

$$10\log_{10}(0.1) = -10 \text{ dB}.$$

So -10 dB is "ten times smaller in power."

5.3 Dynamic Range

The **dynamic range** describes how large the span is between the strongest and weakest relevant components.

Amplitude dynamic range

If amplitudes range from $A_{\min}$ to $A_{\max}$, then the amplitude dynamic range in dB is

$$\Delta_{\text{dB}} = 20\log_{10}\left(\frac{A_{\max}}{A_{\min}}\right).$$

Power dynamic range

If powers range from $P_{\min}$ to $P_{\max}$, then

$$\Delta_{\text{dB}} = 10\log_{10}\left(\frac{P_{\max}}{P_{\min}}\right).$$

Concrete example. Suppose an amplitude ratio is $A_{\max}/A_{\min} = 10^4$. On a linear scale, this is a huge span. In dB,

$$\Delta_{\text{dB}} = 20\log_{10}(10^4) = 20 \cdot 4 = 80 \text{ dB}.$$

So "a factor of 10^4" becomes "80 dB".

5.4 How to Plot Magnitudes in dB (Numerically Safe)

In signal processing we often plot magnitudes such as $|X(\omega)|$, spectral envelopes, or filter frequency responses. Magnitudes can reach extremely small values, including zeros, and $\log(0)$ is undefined. For numerical robustness we use a small offset ε.

Amplitude/magnitude in dB

Given a magnitude $m \geq 0$, define

$$m_{\mathrm{dB}} = 20 \log_{10}(m + \varepsilon),$$

where ε is a tiny constant (typical choice: $\varepsilon = 10^{-12}$).

When to use 20. Use $20 \log_{10}(\cdot)$ for amplitude or magnitude quantities, e.g.:

- $|X(\omega)|$ (magnitude spectrum),
- $|H(\omega)|$ (magnitude response of a filter),
- $|x(t)|$ or RMS amplitude.

Power in dB

For a power-like quantity $p \geq 0$ (e.g. power spectrum, PSD), define

$$p_{\mathrm{dB}} = 10 \log_{10}(p + \varepsilon).$$

Use $10 \log_{10}(\cdot)$ when the quantity is already proportional to $|\cdot|^2$.

Relative dB plots

Often we care about levels relative to a maximum (or to a reference level). For magnitudes, a very common choice is

$$m_{\mathrm{dB,rel}} = 20 \log_{10}\left(\frac{m}{m_{\max}} + \varepsilon\right), \qquad m_{\max} = \max_{\omega} m(\omega).$$

This makes the peak equal to 0 dB, and everything else negative. Such plots are extremely useful for revealing weak components that would be invisible on a linear scale.

5.5 Worked Examples

Example 1: Power ratio to dB and back

Suppose $P_2 = 5P_1$. Then

$$\mathrm{dB} = 10 \log_{10}(5) \approx 10 \cdot 0.6990 \approx 6.99 \text{ dB}.$$

Conversely, if a system has a gain of 6.99 dB in power, then

$$\frac{P_2}{P_1} = 10^{\mathrm{dB}/10} = 10^{0.699} \approx 5.$$

Example 2: Amplitude ratio to dB

Suppose $A_2 = 0.1A_1$. Then

$$\mathrm{dB} = 20\log_{10}(0.1) = 20(-1) = -20 \text{ dB}.$$

So -20 dB corresponds to a tenfold reduction in amplitude.

Example 3: Two tones with very different amplitudes (why dB reveals weak components)

Consider a signal with two sinusoidal components, where the weaker tone has amplitude 100 times smaller:

$$x(t) = \cos(2\pi f_1 t) + 0.01\cos(2\pi f_2 t).$$

The amplitude ratio is 0.01, so the weaker component is

$$20\log_{10}(0.01) = 20(-2) = -40 \text{ dB}$$

below the stronger one. On a linear magnitude plot, the weaker tone may be barely visible (Fig. 25); on a dB plot, it appears clearly at -40 dB relative to the peak (Fig. 26).

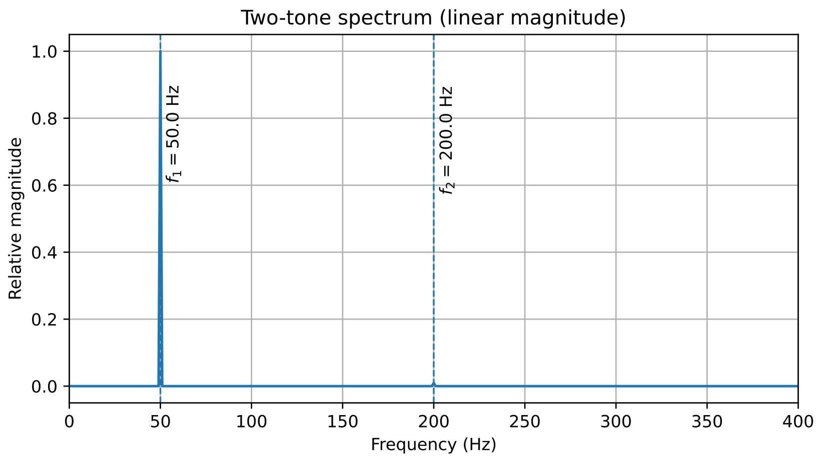

Figure 25: Magnitude spectrum on a linear scale: the weaker tone is hard to see.

6 Plotting Signals with `matplotlib`

Throughout this course, we will frequently visualize signals in order to understand their structure, transformations, and spectral content. In practice, the standard tool for plotting signals in Python is the library `matplotlib`.

This short section provides a practical guide for producing clear signal plots, both in continuous time (CT) and discrete time (DT).

6.1 Continuous-Time Signals: Line Plots

A continuous-time signal is represented numerically by sampling it on a dense time grid:

$$t_0, t_1, \ldots, t_{N-1}, \qquad x(t_i).$$

In `matplotlib`, continuous-time signals are typically plotted using a line plot:

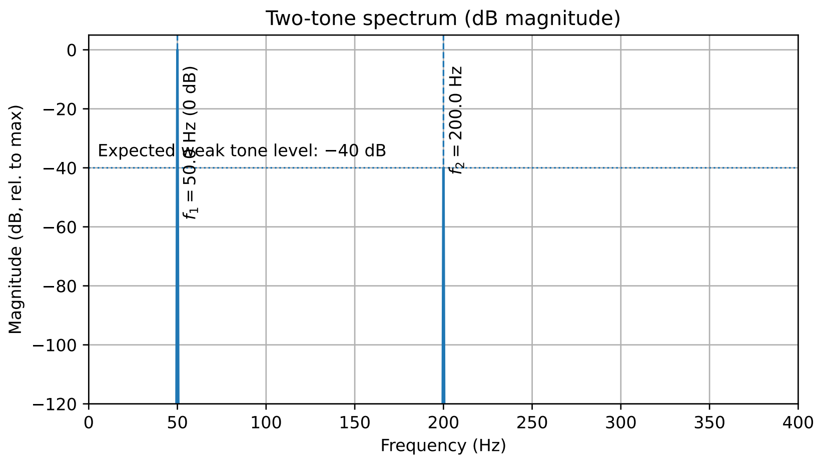

Figure 26: Magnitude spectrum on a dB scale: the weaker tone is clearly visible.

```python
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(-2, 2, 2000)      # time axis
x = np.cos(2*np.pi*3*t)           # example signal

plt.plot(t, x)
plt.grid(True)
plt.xlabel("t")
plt.ylabel("x(t)")
plt.title("Continuous-time signal")
plt.show()
```

Remark. The more points you use in `linspace`, the smoother the curve will appear.

6.2 Discrete-Time Signals: Stem Plots

Discrete-time signals are defined only at integer indices:

$$x[n], \qquad n \in \mathbb{Z}.$$

They should not be plotted as continuous curves. Instead, we use a **stem plot**, which shows individual samples:

```python
n = np.arange(-10, 11)            # index axis
x = (n >= 0).astype(float)        # unit step sequence

plt.stem(n, x, basefmt=" ")
plt.grid(True)
plt.xlabel("n")
plt.ylabel("x[n]")
plt.title("Discrete-time signal")
plt.show()
```

Why stem plots? Because discrete-time signals do not exist "between" integer indices, so connecting points with a line can be misleading.

6.3 Plotting Multiple Signals Together

Often we want to compare an original signal and a transformed version, e.g. a shifted signal:

$$y(t) = x(t - t_0).$$

In `matplotlib`, multiple signals can be plotted on the same figure:

```
t0 = 1.0
y = np.cos(2*np.pi*3*(t - t0))

plt.plot(t, x, label="x(t)")
plt.plot(t, y, label="y(t)=x(t-t0)")
plt.grid(True)
plt.legend()
plt.show()
```

Best practice. Always use legends when plotting more than one signal.

Using Colors in `matplotlib`

When plotting multiple signals together (for example an original signal and its transformed version), using consistent colors greatly improves readability.

Default color cycle. `matplotlib` provides a default sequence of colors labeled:

`C0, C1, C2, C3, ...`

These are convenient because they are visually distinct and work well for scientific plots.

For example:

```
plt.plot(t, x, color="C0", label="x(t)")
plt.plot(t, y, color="C1", label="y(t)")
```

Recommendation. A common convention is:

- `C0` for the original signal,
- `C1, C2, ...` for transformed versions.

Colors for Discrete-Time Stem Plots

For discrete-time signals, we use `stem` plots. Unlike `plot`, the `stem` function requires specifying both line and marker formatting.

Example:

```
plt.stem(n, x,
         linefmt="C0",
         markerfmt="C0o",
         basefmt=" ")
```

Here:

- `linefmt="C0"` sets the stem line color,
- `markerfmt="C0o"` sets the marker color and shape,
- `basefmt=" "` removes the horizontal baseline.

Multiple stem signals. To compare two discrete-time sequences:

```
plt.stem(n, x,
         linefmt="C0",
         markerfmt="C0o",
         label="x[n]")

plt.stem(n, y,
         linefmt="C1",
         markerfmt="C1s",
         label="y[n]")
```

Best practice. Always combine colors with legends:

```
plt.legend()
```

This makes plots much clearer.

6.4 Axis Labels, Titles, and Grid

Good plots should include:

- axis labels (`xlabel`, `ylabel`),
- a descriptive title (`title`),
- a grid (`grid(True)`) (not always).

These features make plots readable and interpretable in reports.

6.5 Saving Figures for Reports

Instead of displaying figures interactively, we often want to save them as PDF files:

```
plt.savefig("my_signal_plot.pdf")
```

A typical workflow is:

```
plt.figure()
plt.plot(t, x)
plt.grid(True)
plt.savefig("signal.pdf")
plt.close()
```

Saving in PDF format is recommended for reports because it preserves vector quality.

6.6 Recommended Style Guidelines

When plotting signals for scientific work:

- Use sufficiently large figure sizes (e.g. `figsize=(6,3)`).
- Keep axis ranges consistent when comparing signals.
- Use clear line widths and markers.
- Avoid clutter: do not overload a single plot with too many signals.

Final remark. Visualization is not only for presentation: plotting signals is an essential tool for developing intuition, debugging computations, and verifying theoretical results. More advanced plotting techniques can be found at `https://github.com/jbmouret/matplotlib_for_papers`; always make sure to look at the documentation of `matplotlib`: `https://matplotlib.org/`.