Signal & Systems

Lecture 6: Discrete Time Fourier Transform

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Template made by Panagiotis Papagiannopoulos

Why do we need the DTFT?

- We want to analyze the **frequency content** of discrete-time signals.
- Many signals of interest are **not periodic**.
- We need a representation that works for **aperiodic discrete-time signals**.

Goal:

Represent a discrete-time signal as a superposition of complex exponentials.

Basic Idea

Recall that complex exponentials

$$e^{j\omega n}$$

play the role of elementary building blocks.

For periodic signals, we used a **discrete set of frequencies** (aka, Fourier Series).

For general aperiodic discrete-time signals, we will need a **continuous frequency variable**:

$$\omega \in \mathbb{R}.$$

This leads to the **Discrete-Time Fourier Transform (DTFT)**.

Definition of the DTFT

For a discrete-time signal $x[n]$, its DTFT is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Interpretation:

- $x[n]$: time-domain signal
- $X(e^{j\omega})$: frequency-domain representation
- ω: angular frequency (continuous variable)

What does the DTFT tell us?

The quantity

$$X(e^{j\omega})$$

describes the contribution of the frequency ω to the signal.

More precisely:

- $|X(e^{j\omega})|$ gives the **strength** of frequency ω
- $\angle X(e^{j\omega})$ gives the **phase** of that component

Idea:

The DTFT measures how strongly $x[n]$ matches the complex exponential $e^{j\omega n}$.

Inverse DTFT

The original signal can be recovered from its DTFT using

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega.$$

Interpretation:

- The signal is synthesized from a continuum of complex exponentials
- The interval $[-\pi, \pi]$ contains all distinct frequencies

Analysis and Synthesis

Forward DTFT (analysis):

$$x[n] \longrightarrow X(e^{j\omega})$$

Break the signal into frequency components.

Inverse DTFT (synthesis):

$$X(e^{j\omega}) \longrightarrow x[n]$$

Combine all frequency components to reconstruct the signal.

Why do we write $X(e^{j\omega})$?

The DTFT is often written as

$$X(e^{j\omega})$$

instead of $X(\omega)$.

This notation emphasizes that the transform is evaluated on the **unit circle** in the complex plane:

$$z = e^{j\omega}.$$

So the DTFT can be viewed as a function of a point on the unit circle.

Important Characteristic: Periodicity

The DTFT is **periodic** in frequency with period 2π.

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

Why?

$$e^{-j(\omega+2\pi)n} = e^{-j\omega n}e^{-j2\pi n} = e^{-j\omega n}$$

since $e^{-j2\pi n} = 1$ for every integer n.

So it is enough to study the DTFT over any interval of length 2π, usually

$$-\pi \leqslant \omega < \pi.$$

Frequency Range

Because of periodicity, the interval

$$-\pi \leqslant \omega < \pi$$

contains all unique frequency information.

Equivalent normalized-frequency form:

$$f = \frac{\omega}{2\pi}, \qquad -\tfrac{1}{2} \leqslant f < \tfrac{1}{2}.$$

Key point:

- Time is discrete
- Frequency is continuous, but periodic

Existence of the DTFT

A common sufficient condition for the DTFT to exist is

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

That is, if $x[n]$ is **absolutely summable**, then the DTFT exists.

Examples:

- $a^n u[n]$, with $|a| < 1$: DTFT exists
- 1 or $e^{j\omega_0 n}$: not absolutely summable

A First Simple Example

Let

$$x[n] = \delta[n].$$

Then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1.$$

Interpretation:

- The unit impulse contains all frequencies equally
- Its DTFT is flat

Pair 1: Unit impulse

Let

$$x[n] = \delta[n].$$

Then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1.$$

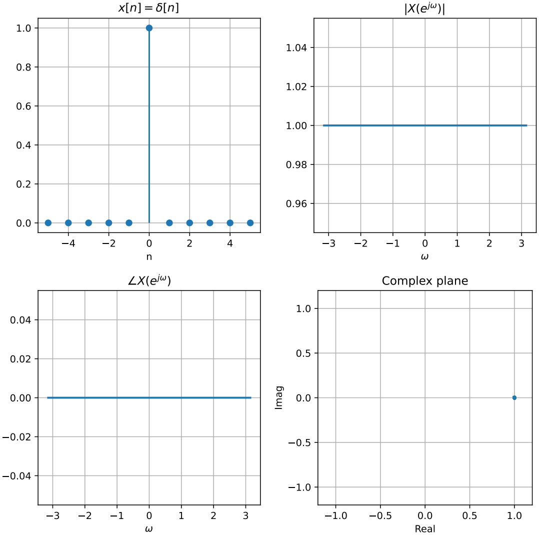
So the pair is

$$\boxed{\delta[n] \longleftrightarrow 1}$$

Interpretation:

- The impulse contains all frequencies equally.
- Its spectrum is flat.

Pair 1: Visualization



Pair 2: Shifted impulse

Let

$$x[n] = \delta[n - n_0].$$

Then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}.$$

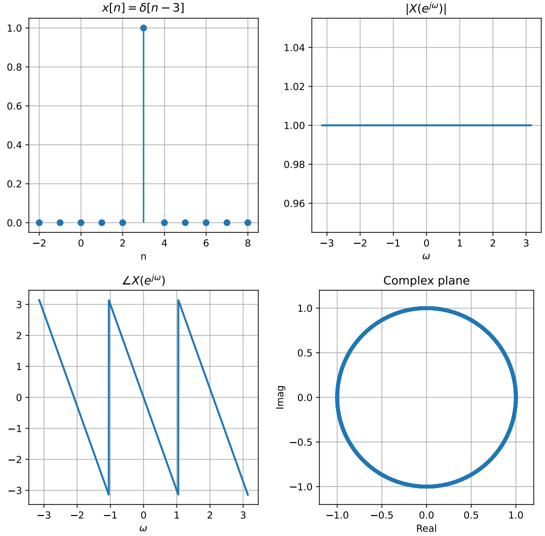
Hence

$$\boxed{\delta[n - n_0] \;\longleftrightarrow\; e^{-j\omega n_0}}$$

Interpretation:

- A time shift produces a linear phase term.
- Magnitude stays equal to 1 for all ω.

Pair 2: Visualization



Pair 3: Constant sequence

Consider

$$x[n] = 1.$$

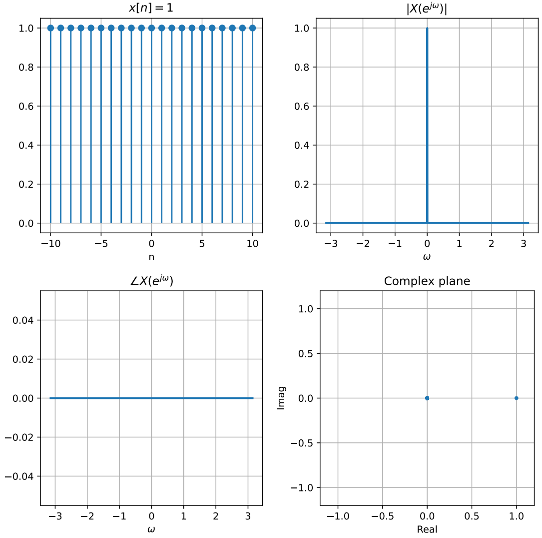
This is not absolutely summable, so its DTFT is interpreted in the generalized sense:

$$\boxed{1 \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)}$$

Interpretation:

- A constant sequence has only a DC component.
- Because the DTFT is 2π-periodic, the impulse repeats every 2π.

Pair 3: Visualization



What is the DC Component?

Consider a constant (time-invariant) signal:

$$x[n] = C.$$

Its DTFT is

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k).$$

Definition:

- The **DC component** is the part of the signal at $\omega = 0$.
- It corresponds to the **average value** of the signal.

For general signals:

$$\text{DC value} = X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n].$$

Why is it called DC?

DC stands for **Direct Current** (from electrical engineering).

Analogy:

- DC signal: constant voltage/current over time
- AC signal: oscillating signal (sinusoids)

In signals:

- DC component → constant (zero frequency)
- AC components → oscillations (nonzero frequency)

Key idea:

Frequency $\omega = 0$ means **no oscillation**.

So a constant signal is the **zero-frequency component**.

Pair 4: Complex exponential

Let

$$x[n] = e^{j\omega_0 n}.$$

Its DTFT is

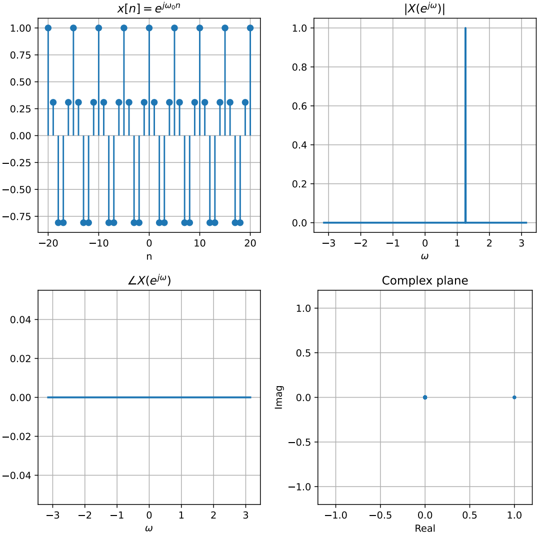
$$\boxed{e^{j\omega_0 n} \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)}$$

Interpretation:

- A pure complex exponential corresponds to impulses in frequency.
- The spectrum repeats every 2π.

This is the discrete-time analogue of a pure tone.

Pair 4: Visualization



Pair 5: Real cosine

Using

$$\cos(\omega_0 n) = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n},$$

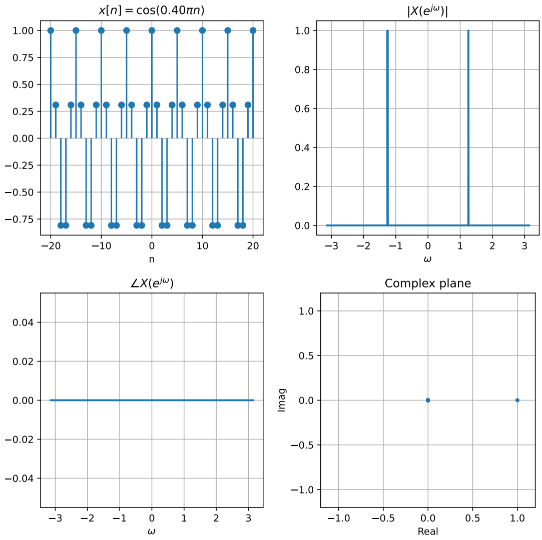
we get

$$\boxed{\cos(\omega_0 n) \longleftrightarrow \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi k)}$$

Interpretation:

- A real cosine has two symmetric frequency components.
- Real signals often produce conjugate-symmetric spectra.

Pair 5: Visualization



Pair 6: Right-sided decaying exponential

Let

$$x[n] = a^n u[n], \qquad |a| < 1.$$

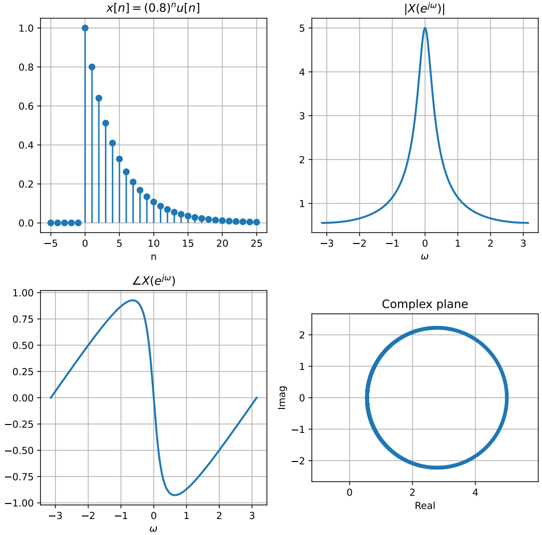
Then

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n.$$

Since $|a| < 1$, this geometric series gives

$$\boxed{a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\omega}}, \qquad |a| < 1}$$

Pair 6: Visualization



Decaying exponential: interpretation

For

$$x[n] = a^n u[n], \qquad |a| < 1,$$

the DTFT is

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

Magnitude:

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}.$$

Interpretation:

- Slow decay in time $\Rightarrow$ sharper frequency concentration.
- Faster decay in time $\Rightarrow$ broader spectrum.

Pair 7: Finite rectangular pulse

Let

$$x[n] = \begin{cases} 1, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

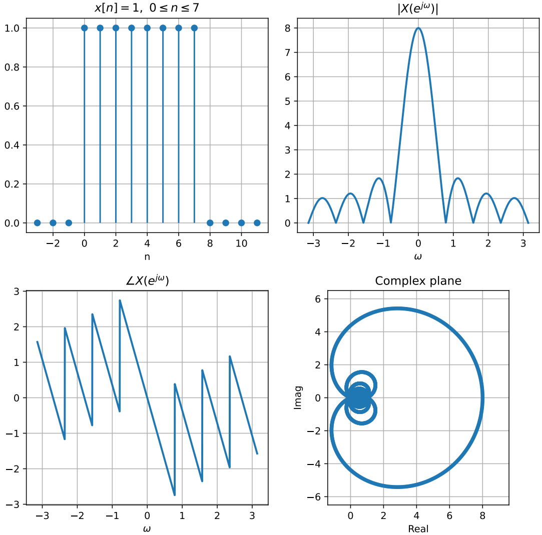
Then

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n}.$$

Evaluating the geometric sum:

$$\boxed{X(e^{j\omega}) = e^{-j\omega(N-1)/2} \frac{\sin(N\omega/2)}{\sin(\omega/2)}}$$

Pair 7: Visualization



Rectangular pulse: interpretation

For the finite-length pulse,

$$X(e^{j\omega}) = e^{-j\omega(N-1)/2}\frac{\sin(N\omega/2)}{\sin(\omega/2)}.$$

What to notice:

- The spectrum is continuous in ω.
- The magnitude has a main lobe and side lobes.
- Increasing N narrows the main lobe.

Rectangular pulse: interpretation

For the finite-length pulse,

$$X(e^{j\omega}) = e^{-j\omega(N-1)/2}\frac{\sin(N\omega/2)}{\sin(\omega/2)}.$$

What to notice:

- The spectrum is continuous in ω.
- The magnitude has a main lobe and side lobes.
- Increasing N narrows the main lobe.

Big lesson:

Finite duration in time produces spread in frequency.

A useful summary table

Signal $x[n]$	**DTFT** $X(e^{j\omega})$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$e^{j\omega_0 n}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$
$a^n u[n],\ \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$\mathbf{1}_{[0,N-1]}[n]$	$e^{-j\omega(N-1)/2} \frac{\sin(N\omega/2)}{\sin(\omega/2)}$

Key takeaways

- Impulses in time correspond to flat spectra.
- Pure exponentials in time correspond to impulses in frequency.
- Decay in time leads to smooth spectra.
- Finite support in time leads to spread in frequency.

DTFT Properties: Overview

If

$$x[n] \longleftrightarrow X(e^{j\omega}),$$

then we can derive many transforms using properties instead of direct computation.

Main properties:

- Linearity
- Time shift
- Frequency shift (modulation)
- Conjugation and symmetry
- Time reversal
- Convolution
- Multiplication

Linearity

If

$$x_1[n] \leftrightarrow X_1(e^{j\omega}), \quad x_2[n] \leftrightarrow X_2(e^{j\omega}),$$

then

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega}).$$

Interpretation:

- The DTFT is a linear transform.
- Superposition holds in both domains.

Time Shift

If

$$x[n] \leftrightarrow X(e^{j\omega}),$$

then

$$x[n - n_0] \longleftrightarrow X(e^{j\omega})\, e^{-j\omega n_0}.$$

Interpretation:

- A shift in time introduces a **linear phase**.
- Magnitude is unchanged.

Frequency Shift (Modulation)

If

$$x[n] \leftrightarrow X(e^{j\omega}),$$

then

$$x[n]e^{j\omega_0 n} \longleftrightarrow X(e^{j(\omega-\omega_0)}).$$

Interpretation:

- Multiplication by a complex exponential shifts the spectrum.
- Used in modulation and communications.

Time Reversal

If

$$x[n] \leftrightarrow X(e^{j\omega}),$$

then

$$x[-n] \longleftrightarrow X(e^{-j\omega}).$$

Interpretation:

- Time reversal flips the spectrum.

Conjugation

If

$$x[n] \leftrightarrow X(e^{j\omega}),$$

then

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega}).$$

Special case: real signals

$$x[n] \in \mathbb{R} \;\Rightarrow\; X(e^{j\omega}) = X^*(e^{-j\omega})$$

Interpretation:

- Real signals produce conjugate-symmetric spectra.

Even/Odd Symmetry

For real signals:

$$\text{Re}\{X(e^{j\omega})\} \text{ is even,} \quad \text{Im}\{X(e^{j\omega})\} \text{ is odd.}$$

Implications:

- Magnitude is even:

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

- Phase is odd (up to wrapping).

Convolution in Time

If

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k],$$

then

$$x[n] * h[n] \longleftrightarrow X(e^{j\omega})H(e^{j\omega}).$$

Interpretation:

- Convolution in time becomes multiplication in frequency.
- Fundamental for LTI systems.

Multiplication in Time

If

$$x[n] \cdot h[n] \longleftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta,$$

then

$$x[n]h[n] \longleftrightarrow \frac{1}{2\pi} (X * H)(\omega).$$

Interpretation:

- Multiplication in time becomes convolution in frequency.
- Dual to the previous property.

Difference Operator (Discrete Gradient)

Define the first difference (discrete derivative):

$$y[n] = x[n] - x[n-1].$$

Using time-shift property:

$$x[n-1] \longleftrightarrow X(e^{j\omega})e^{-j\omega},$$

we get

$$y[n] \longleftrightarrow X(e^{j\omega})\left(1 - e^{-j\omega}\right).$$

Key result:

$$\boxed{x[n] - x[n-1] \longleftrightarrow X(e^{j\omega})(1 - e^{-j\omega})}$$

Interpretation:

- Acts like a **high-pass filter**.
- For small ω:

  $$1 - e^{-j\omega} \approx j\omega,$$

  so it approximates a derivative.
- Suppresses low frequencies (DC $\rightarrow$ 0).

Summary

- Time shift → phase shift
- Modulation → frequency shift
- Convolution → multiplication
- Multiplication → convolution
- Real signals → conjugate symmetry

Big idea:

The DTFT converts difficult time-domain operations into simpler frequency-domain operations.

DTFT Summary

- The DTFT of $x[n]$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

- The inverse DTFT is

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega.$$

- The DTFT is:
  - continuous in ω
  - periodic with period 2π
- It provides the frequency-domain description of a discrete-time signal.

From Infinite to Finite Signals

So far, we assumed signals exist for all n.

In practice, we observe only:

$$x[n], \quad 0 \leqslant n \leqslant N-1.$$

This is equivalent to:

$$x_{\text{obs}}[n] = x[n] \cdot w[n],$$

where

$$w[n] = \begin{cases} 1, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

Key question:

How does finite duration affect the spectrum?

Example 1: Finite Rectangular Pulse

Let

$$x[n] = \begin{cases} 1, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{otherwise} \end{cases}$$

DTFT:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n}.$$

Evaluating the sum:

$$X(e^{j\omega}) = e^{-j\omega(N-1)/2} \frac{\sin(N\omega/2)}{\sin(\omega/2)}.$$

This is the **Dirichlet kernel**.

Rectangular Pulse: Spectrum Shape

Magnitude:

$$|X(e^{j\omega})| = \left|\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right|.$$

Key features:

- **Main lobe** centered at $\omega = 0$
- **Side lobes** decay away from center
- Zeros at:

$$\omega = \frac{2\pi k}{N}, \quad k \neq 0$$

Interpretation:

Finite duration $\Rightarrow$ spread in frequency

Effect of Increasing N

For the rectangular pulse:

As N increases:

- Main lobe becomes **narrower**
- Side lobes become more **dense**
- Spectrum becomes more **concentrated**

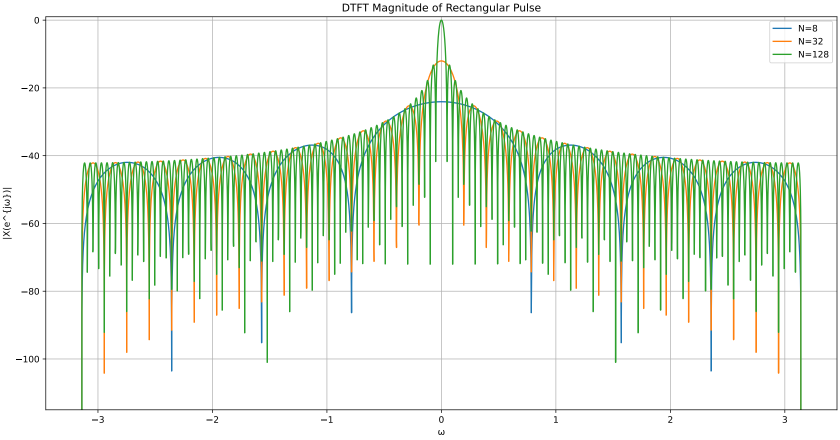
Approximate main-lobe width:

$$\Delta\omega \approx \frac{4\pi}{N}.$$

Key idea:

Longer signal $\Rightarrow$ better frequency resolution

Effect of Increasing N: Visualization



Example 2: Finite Sinusoid

Let

$$x[n] = \cos(\omega_0 n), \quad 0 \leqslant n \leqslant N-1.$$

Using Euler:

$$\cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}.$$

Each term produces a shifted Dirichlet kernel:

$$X(e^{j\omega}) = \frac{1}{2} D_N(\omega - \omega_0) + \frac{1}{2} D_N(\omega + \omega_0).$$

Finite Sinusoid: Spectral Leakage

For an infinite sinusoid:

$$\cos(\omega_0 n) \longleftrightarrow \text{impulses at } \pm\omega_0.$$

For a finite sinusoid:

$$\text{impulses } \Rightarrow \text{spread lobes}$$

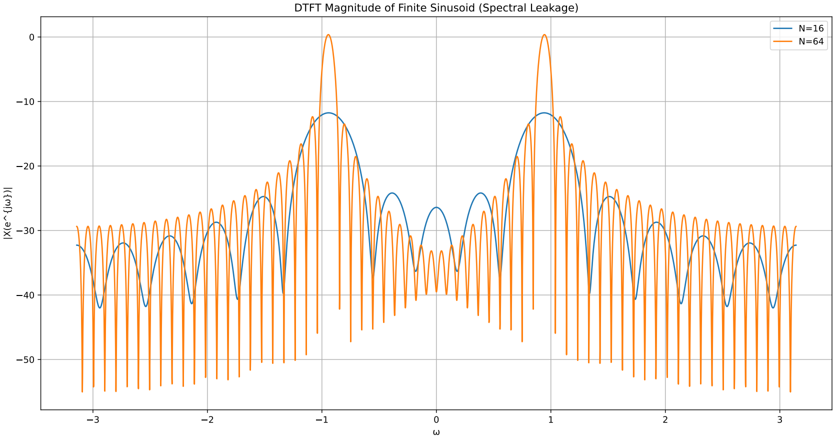
This is called:

Spectral leakage

Interpretation:

- Energy spreads around ω_0
- No longer perfectly localized

Finite Sinusoid: Visualization



Why Does Leakage Happen?

We observe:

$$x_{\text{obs}}[n] = x[n] \cdot w[n].$$

Multiplication in time $\Rightarrow$ convolution in frequency:

$$X_{\text{obs}}(e^{j\omega}) = X(e^{j\omega}) * W(e^{j\omega}).$$

Since $W(e^{j\omega})$ is the Dirichlet kernel:

Impulses get **smeared into lobes**

Key insight:

Windowing causes spectral spreading

Summary: Finite-Length Effects

- Finite signals $\Rightarrow$ continuous spectra
- Impulses $\Rightarrow$ Dirichlet kernels
- Main lobe $\rightarrow$ resolution
- Side lobes $\rightarrow$ leakage

Big idea:

Time limitation introduces frequency spreading

Thank you

- **Any Questions?**
- **Office Hours**:
  - **Mon & Tue** *(09:00-11:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_SS_AM*)
- **Material and Announcements**

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