Signal & Systems

Lecture 4: Fourier Transform

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Template made by Panagiotis Papagiannopoulos

How should we represent signals?

- A signal is information as a function of time

- We want a representation that is:
    - **compact**
    - **reveals structure**
    - **makes system analysis easier**

$$\text{Goal: } x(t) \longrightarrow \text{useful representation}$$

Today: we change our viewpoint from time-domain to frequency-domain representations.

Reminder: vectors and bases

- Any vector can be written as a combination of basis vectors
- Coefficients are **projections**
- Orthogonality makes computations easy

$$\mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$$

Signals as infinite-dimensional vectors

$$x(t) = \sum_k a_k\,\phi_k(t)$$

Choose basis functions $\{\phi_k(t)\}$ wisely.

Why choose sinusoids as basis?

- Sinusoids are **eigenfunctions** of LTI systems
- They do **not change shape** through an LTI system
- Only amplitude and phase change

$$x(t) = e^{j\omega t} \quad \implies \quad y(t) = H(\omega)\, e^{j\omega t}$$

Key idea

Sinusoids "diagonalize" LTI systems.

Time domain

- **When** something happens
- Local behavior (transients, edges)
- Convolution describes LTI outputs

Frequency domain

- **What frequencies** are present
- Global structure (oscillations)
- Multiplication replaces convolution

$$x(t) \text{ (time)} \quad \Longleftrightarrow \quad X(\omega) \text{ (frequency)}$$

Fourier analysis: the idea

- Represent signals by projecting onto complex exponentials
- Periodic signals → **Fourier Series**
- General (aperiodic) signals → **Fourier Transform**

Periodic: Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}$$

Aperiodic: Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Periodic signals

- A continuous-time signal $x(t)$ is **periodic** with period $T > 0$ if

$$x(t + T) = x(t) \quad \forall t$$

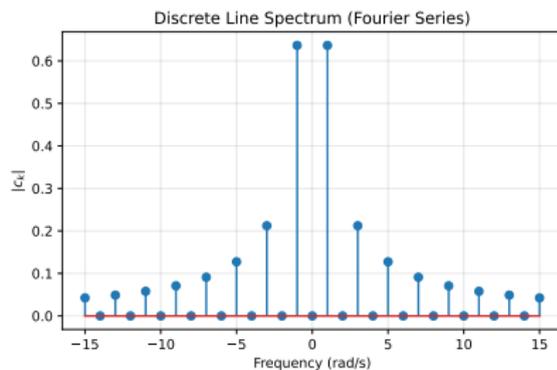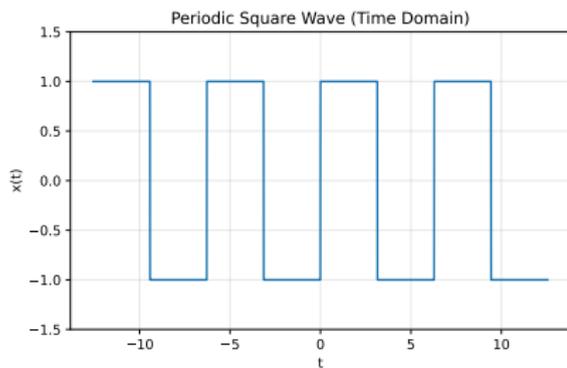- Fundamental (angular) frequency:

$$\omega_0 = \frac{2\pi}{T}$$

- Examples: sinusoids, square/triangle waves, periodic pulse trains

Key idea

Periodic signals have a **discrete line spectrum** at harmonics $\omega = k\omega_0$.

Periodic Square Wave (Time Domain)

Discrete Line Spectrum (Fourier Series)

Why complex exponentials?

- Use the basis functions

$$\phi_k(t) = e^{jk\omega_0 t}, \quad k \in \mathbb{Z}$$

- They are **periodic** with period T and form an **orthogonal** set over any length-T interval

Orthogonality (over one period)
$$\frac{1}{T} \int_{t_0}^{t_0+T} e^{jk\omega_0 t} e^{-jm\omega_0 t} \, dt = \delta_{k,m}$$

$\delta_{k,m}$ is the Kronecker delta: 1 if $k = m$, 0 otherwise.

Fourier Series (complex form)

Representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t}$$

- $\{c_k\}$ are complex coefficients (amplitude + phase information)
- Each term corresponds to a harmonic frequency $k\omega_0$

Interpretation

$x(t)$ is decomposed into a **weighted sum of harmonically related sinusoids**.

Fourier Series coefficients

$$c_k = \frac{1}{T} \int_{t_0}^{t_0 + T} x(t)\, e^{-jk\omega_0 t}\, dt$$

- This is an **inner product** (projection) onto the basis function $e^{jk\omega_0 t}$
- Orthogonality guarantees:
  - coefficients are uniquely determined (under standard conditions)
  - "no mixing" between different harmonics

$$\langle x, \phi_k \rangle \triangleq \frac{1}{T} \int_{t_0}^{t_0 + T} x(t)\phi_k^*(t)\, dt$$

where $\phi_k(t) = e^{jk\omega_0 t}$.

Signal inner product (over one period)

$$\langle x, y \rangle \;=\; \frac{1}{T} \int_{t_0}^{t_0 + T} x(t)\, y^*(t)\, dt$$

- This satisfies the properties of an inner product:
  - Linearity
  - Conjugate symmetry
  - Positive definiteness
- The Fourier coefficient is a projection:

$$c_k = \langle x(t), e^{jk\omega_0 t} \rangle$$

Exactly analogous to $c_k = \mathbf{x}^\top \mathbf{e}_k$ *in linear algebra.*

Start from the complex form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

- Assume $x(t)$ is **real-valued**
- Then the coefficients satisfy the conjugate symmetry property:

$$c_{-k} = c_k^*$$

Why?

If $x(t)$ is real, its complex exponential terms must pair up to produce a real signal.

Rewrite the series:

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right)$$

Using $c_{-k} = c_k^*$:

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right)$$

Key idea

Positive and negative frequencies form conjugate pairs.

Let

$$c_k = \alpha_k + j\beta_k$$

Using Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

We obtain:

$$c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} = 2\Re\{c_k\}\cos(k\omega_0 t) - 2\Im\{c_k\}\sin(k\omega_0 t)$$

Important

Each conjugate pair produces one cosine and one sine.

Define:

$$a_k = 2\Re\{c_k\}, \qquad b_k = -2\Im\{c_k\}$$

and

$$a_0 = c_0$$

Then the series becomes:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right]$$

Conclusion

The sine/cosine form is just the real-valued version of the complex Fourier Series.

Discrete spectrum (line spectrum)

Fourier Series gives a **discrete** set of frequency components:

$$\omega = k\omega_0, \quad k \in \mathbb{Z}$$

- Magnitude: $|c_k|$ tells "how much" of harmonic k
- Phase: $\angle c_k$ tells alignment / timing information

Takeaway
Fourier Series $\Rightarrow$ periodic signals $\leftrightarrow$ **discrete** frequency representation.

Let $S_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}$ be the N-term partial sum.

- For many signals, $S_N(t) \to x(t)$ as $N \to \infty$ (pointwise except at discontinuities)
- At a jump discontinuity:
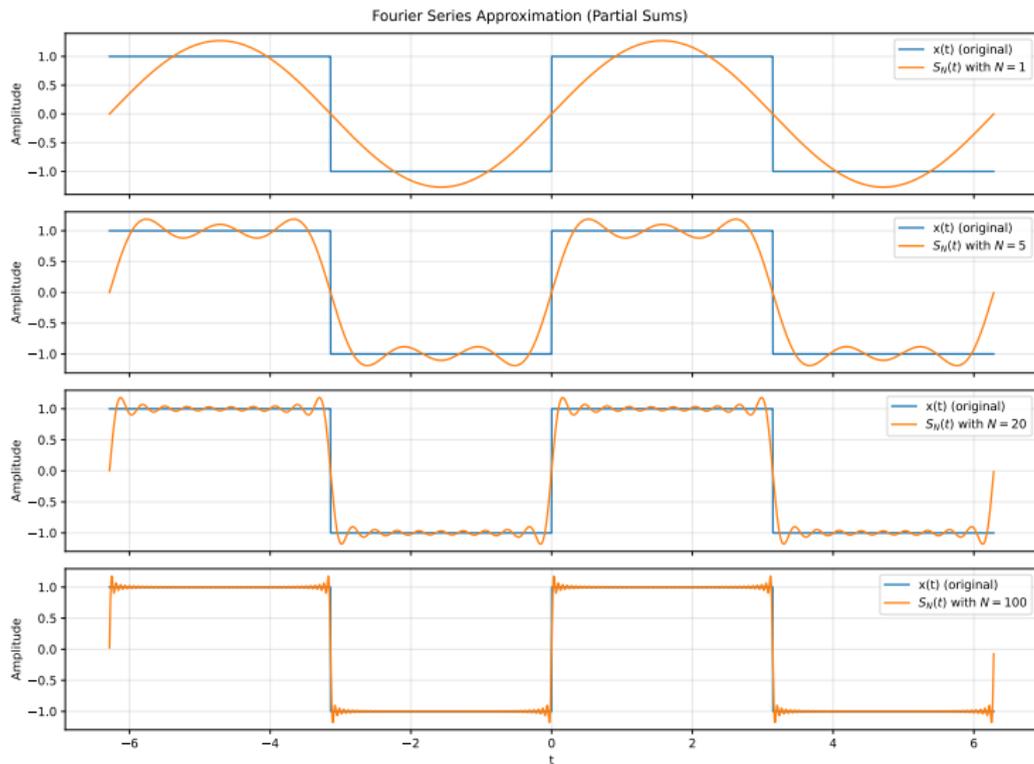
$$S_N(t) \to \frac{x(t^-) + x(t^+)}{2}$$

- Near discontinuities: **ringing** and **overshoot** (Gibbs)

Why it matters

FS is a powerful model, but it has limitations.

Fourier Series Approximation (Partial Sums)

Discrete frequencies

For a periodic signal with period T:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where

$$\omega_0 = \frac{2\pi}{T}$$

Therefore the signal contains frequencies

$$\omega = k\omega_0$$

- Only **harmonics of the fundamental frequency** appear
- The spectrum is **discrete**

Discrete frequencies

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Therefore the signal contains frequencies

$$\omega = k\omega_0$$

- Only **harmonics of the fundamental frequency** appear
- The spectrum is **discrete**

What happens when the period becomes very large? $T \to \infty$?

Recall

$$\omega_0 = \frac{2\pi}{T}$$

- If T increases, the spacing between spectral lines decreases

$$\omega_0 \downarrow$$

When the period grows

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- Frequencies become more densely packed

When the period grows

Recall

$$\omega_0 = \frac{2\pi}{T}$$

- If T increases, the spacing between spectral lines decreases

$$\omega_0 \downarrow$$

- Frequencies become more densely packed
- In the limit

$$T \to \infty$$

the frequency spacing goes to zero

$$\omega_0 \to 0$$

Result

The discrete spectrum becomes a **continuous spectrum**.

From sums to integrals

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

As $T \to \infty$

$$\omega_0 \to 0$$

Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$
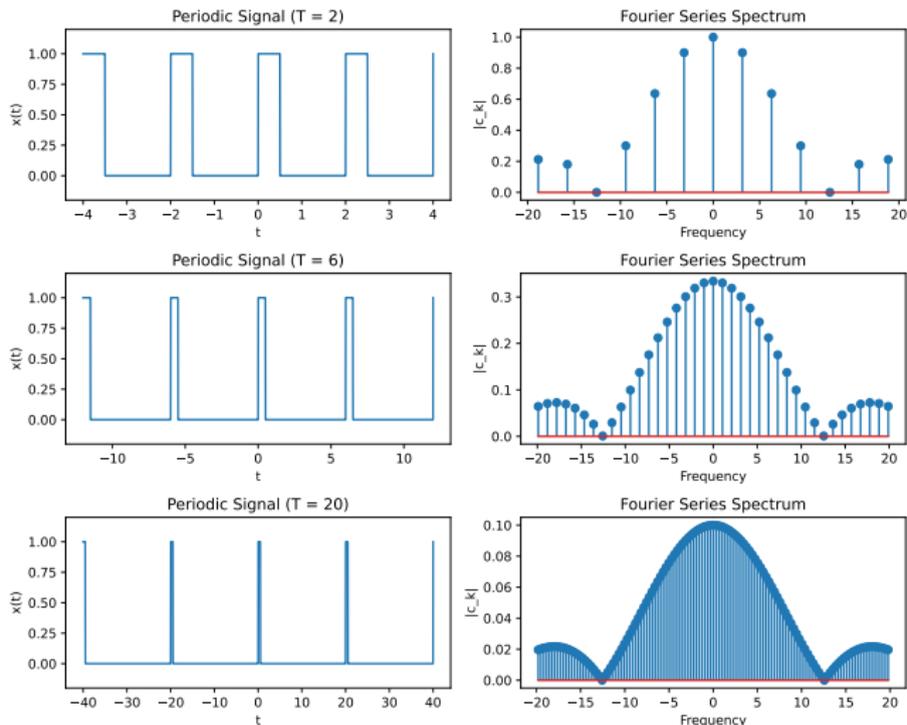
As $T \to \infty$

$$\omega_0 \to 0$$

The sum becomes an integral:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier Transform

A representation for **non-periodic signals**.

Fourier Series vs Fourier Transform

Fourier Series	Fourier Transform
Periodic signal	Aperiodic signal
Discrete frequencies ($k\omega_0$)	Continuous frequencies (ω)
Coefficients c_k	Spectrum $X(\omega)$
$x(t) = \sum_k c_k e^{jk\omega_0 t}$	$x(t) = \dfrac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega$

Forward Fourier Transform

For a signal $x(t)$ (possibly complex), its Fourier Transform is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Inverse Fourier Transform

The original signal can be recovered from its spectrum:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

- $x(t)$: time-domain signal
- $X(\omega)$: frequency-domain representation (spectrum)
- ω: angular frequency (rad/s)

Definition

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

- Many useful transforms appear repeatedly
- We often reason using **known transform pairs**

Idea

If we know the transform of a signal, we automatically know the inverse relationship.

Transform Pair: Impulse and Constant

$$\delta(t) \quad \longleftrightarrow \quad 1$$

$$1 \quad \longleftrightarrow \quad 2\pi\delta(\omega)$$

- An impulse in time contains **all frequencies equally**
- A constant signal contains **only zero frequency**

Physical meaning

Perfect localization in time $\Rightarrow$ complete spread in frequency.

Example: Impulse Signal

Consider the impulse signal

$$x(t) = \delta(t)$$

Compute its Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

Using the sifting property of the impulse:

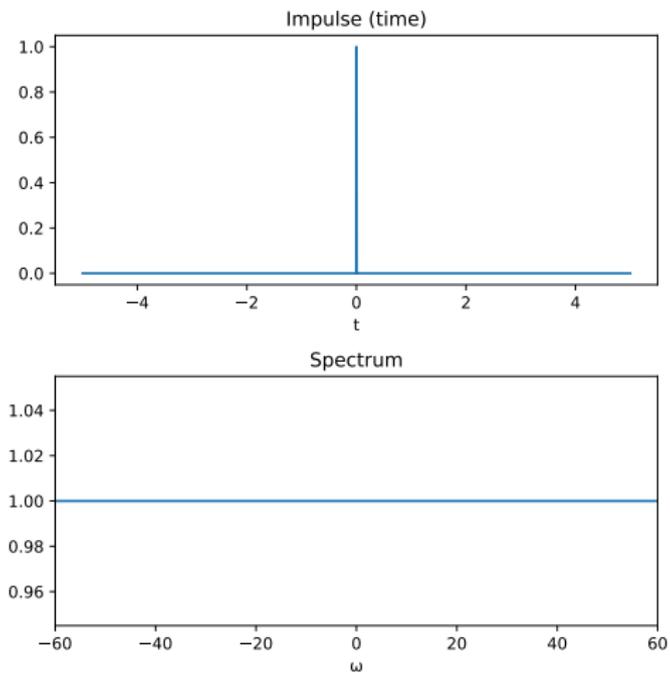$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

we obtain

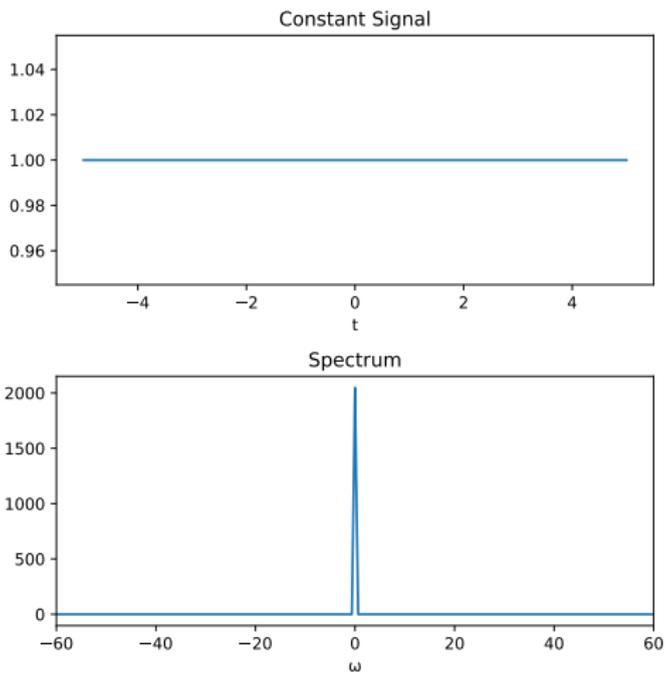$$X(\omega) = e^{-j\omega \cdot 0} = 1$$

Result

$$\delta(t) \longleftrightarrow 1$$

$$\text{rect}\left(\frac{t}{T}\right) \quad \longleftrightarrow \quad T \, \text{sinc}\left(\frac{\omega T}{2}\right)$$

Short pulse in time $\rightarrow$ wide spectrum

Key observation

Sharper time localization produces a broader frequency spectrum.

Consider the signal

$$x(t) = \begin{cases} 1, & |t| \leqslant \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Compute its Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Since the signal is nonzero only on $[-T/2, T/2]$:

$$X(\omega) = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

Compute the integral:

$$X(\omega) = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T/2}^{T/2}$$

$$= \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$
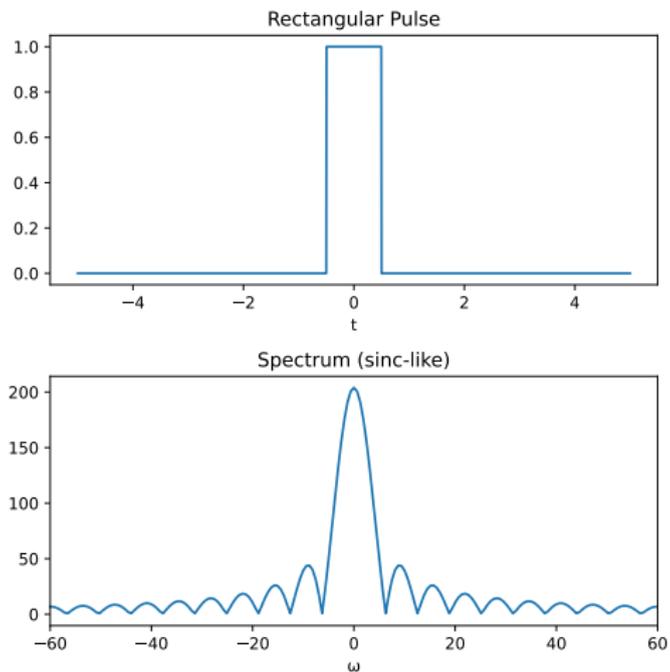
Using

$$e^{j\theta} - e^{-j\theta} = 2j \sin\theta$$

we obtain

$$X(\omega) = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

Consider

$$x(t) = e^{j\omega_0 t}$$

Compute the Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Substitute $x(t)$:

$$X(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt$$

Combine exponents:

$$X(\omega) = \int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} dt$$

Consider

$$X(\omega) = \int_{-\infty}^{\infty} e^{-j(\omega - \omega_0)t} dt$$

This integral does not converge in the usual sense.
Instead it is interpreted in the sense of generalized functions:

$$\int_{-\infty}^{\infty} e^{-j\alpha t} dt = 2\pi\delta(\alpha)$$

Applying this identity with $\alpha = \omega - \omega_0$:
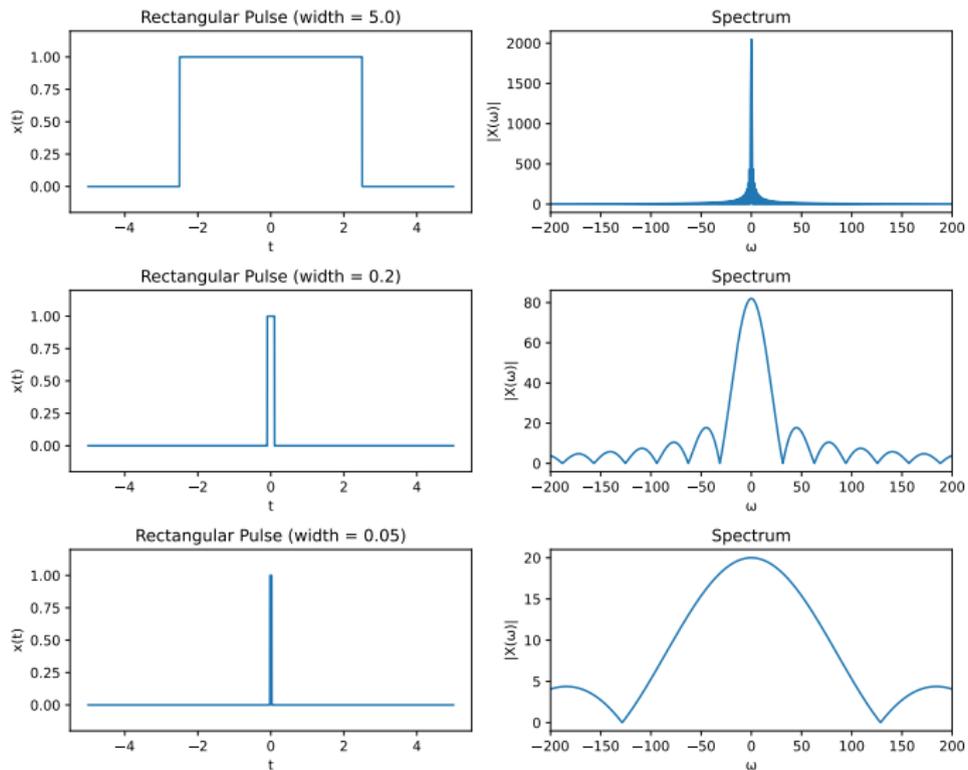
$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

Time domain	Frequency domain
Short signal	Wide spectrum
Long signal	Narrow spectrum
Impulse	All frequencies
Constant/Periodic signal	Single frequency

Fundamental principle

Signals cannot be simultaneously localized in time and frequency.

Time-Frequency Tradeoff: Visualization

If

$$x(t) \longleftrightarrow X(\omega)$$

then several useful relationships follow.

- Linearity
- Time shifting
- Modulation (frequency shift)
- Convolution

Why these matter

They allow us to compute Fourier transforms and analyze systems without directly evaluating integrals.

Property 1: Linearity

If

$$x_1(t) \leftrightarrow X_1(\omega) \qquad x_2(t) \leftrightarrow X_2(\omega)$$

then

$$a\,x_1(t) + b\,x_2(t) \;\longleftrightarrow\; a\,X_1(\omega) + b\,X_2(\omega)$$

Reason

The Fourier Transform is defined by an integral, which is linear.

Interpretation

The spectrum of a sum of signals is the sum of their spectra.

Property 2: Time Shift

If

$$x(t) \leftrightarrow X(\omega)$$

then

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$$

Interpretation

A time delay does not change the magnitude of the spectrum, but introduces a phase shift.

$$|X(\omega)| \text{ unchanged}$$

$$\angle X(\omega) \to \angle X(\omega) - \omega t_0$$

Property 3: Modulation

If

$$x(t) \leftrightarrow X(\omega)$$

then

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$$

Interpretation

Multiplying a signal by a complex sinusoid shifts its spectrum.

This is the basis of modulation in communication systems.

Using Euler's identity

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

we obtain

$$x(t)\cos(\omega_0 t) \longleftrightarrow \frac{1}{2}\Big[X(\omega - \omega_0) + X(\omega + \omega_0)\Big]$$

Property 3: Modulation (2)

Using Euler's identity

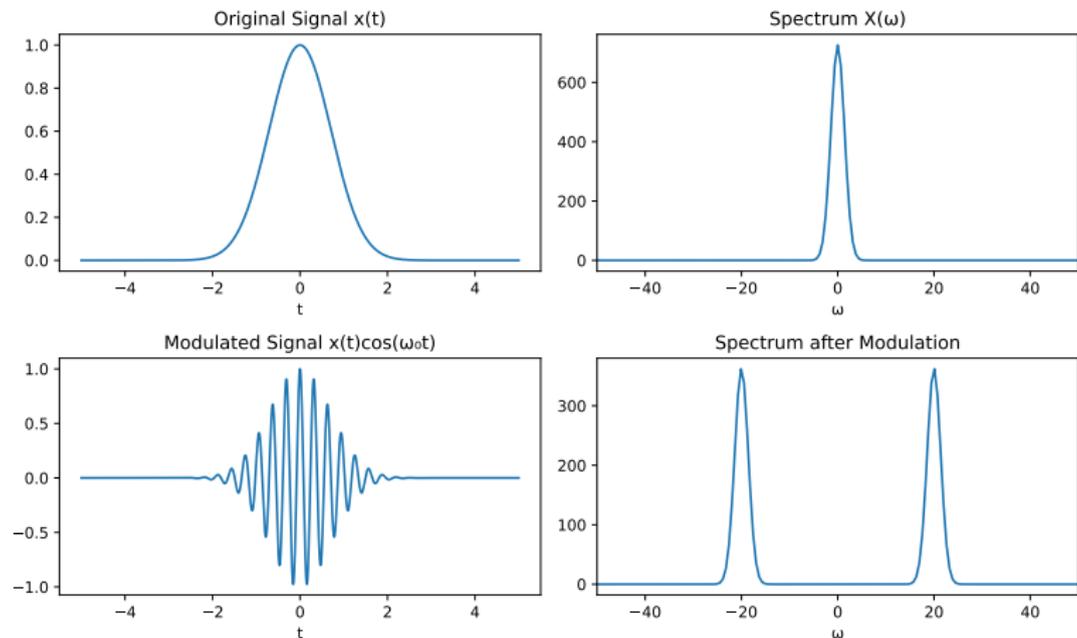$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

we obtain

$$x(t)\cos(\omega_0 t) \;\longleftrightarrow\; \frac{1}{2}\Big[X(\omega - \omega_0) + X(\omega + \omega_0)\Big]$$

Similarly,

$$x(t)\sin(\omega_0 t) \;\longleftrightarrow\; \frac{1}{2j}\Big[X(\omega - \omega_0) - X(\omega + \omega_0)\Big]$$

Why Do We Need Modulation?

- Many real-world signals (audio, sensors, speech) are **low-frequency** signals.
- However, wireless communication systems operate at **high frequencies**.

Example: AM Radio

- Voice signal: $\approx$ 300 Hz—3 kHz
- Radio carrier: $\approx$ 1 MHz

To transmit the signal, we multiply it by a high-frequency carrier:

$$s(t) = x(t)\cos(\omega_c t)$$

This shifts the spectrum to the carrier frequency.

Result

- The signal can be transmitted efficiently through the air
- Different users can occupy different frequency bands

Property 4: Convolution

If

$$x(t) \leftrightarrow X(\omega) \qquad h(t) \leftrightarrow H(\omega)$$

then

$$x(t) * h(t) \longleftrightarrow X(\omega)H(\omega)$$

and

$$x(t)h(t) \longleftrightarrow \frac{1}{2\pi}X(\omega) * H(\omega)$$

Key result

Convolution in time becomes multiplication in frequency. Multiplication in time becomes convolution in frequency.

Additional Fourier Transform Properties

If

$$x(t) \longleftrightarrow X(\omega)$$

then the following useful properties also hold:

- **Time Scaling**

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

- **Differentiation in Time**

$$\frac{d}{dt} x(t) \longleftrightarrow j\omega X(\omega)$$

- **Integration in Time**

$$\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{1}{j\omega} X(\omega)$$

- **Parseval's Theorem**

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- **Duality**

$$x(t) \longleftrightarrow X(\omega) \quad \Rightarrow \quad X(t) \longleftrightarrow 2\pi x(-\omega)$$

Any Fourier Transform can be written as

$$X(\omega) = |X(\omega)|e^{j\angle X(\omega)}$$

- $|X(\omega)|$: **Magnitude spectrum**
- $\angle X(\omega)$: **Phase spectrum**

Interpretation

- Magnitude tells **how much** of each frequency exists
- Phase tells **how those frequencies align in time**

Computing the Phase of a Spectrum

The Fourier Transform is generally complex:

$$X(\omega) = \Re\{X(\omega)\} + j\,\Im\{X(\omega)\}$$

We can express it in polar form:

$$X(\omega) = |X(\omega)|e^{j\angle X(\omega)}$$

Magnitude

$$|X(\omega)| = \sqrt{(\Re\{X(\omega)\})^2 + (\Im\{X(\omega)\})^2}$$

Phase

$$\angle X(\omega) = \arctan\left(\frac{\Im\{X(\omega)\}}{\Re\{X(\omega)\}}\right)$$

In practice

Use the function $\mathrm{atan2}(\Im, \Re)$ to obtain the correct phase quadrant.

- The magnitude spectrum indicates the **strength of each frequency**

$$|X(\omega)|$$

Large value $\Rightarrow$ strong presence of that frequency

Example

A signal with sharp transitions requires many high frequencies.

$$\angle X(\omega)$$

- Phase determines how frequency components align in time
- Changing phase can shift or reshape the signal

Example

A time shift produces a linear phase shift:

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$$

Suppose we reconstruct a signal using:

- Correct magnitude but random phase
- Correct phase but incorrect magnitude

Why Phase Matters

Suppose we reconstruct a signal using:

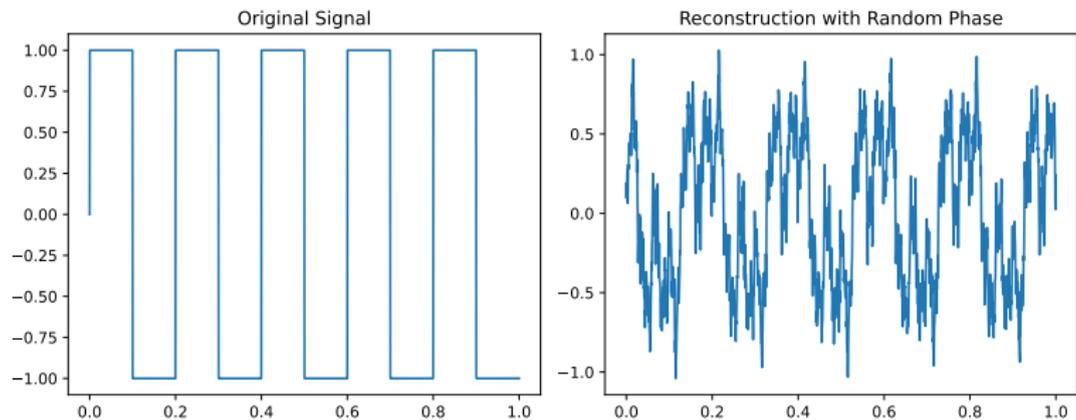- Correct magnitude but random phase
- Correct phase but incorrect magnitude

Observation

Phase often carries much of the signal structure.

> Magnitude: **how much**
> Phase: **where**

Image 1

Image 2

Magnitude 1 + Phase 2

Magnitude 2 + Phase 1

Thank you

- **Any Questions?**

- **Office Hours**:
    - **Mon & Tue** *(09:00-11:00)*

    - 24/7 by email (costashatz@upatras.gr, subject: *ECE_SS_AM*)

- **Material and Announcements**

Laboratory of Automation & Robotics