Signal & Systems

Lecture 3: Convolution and Correlation

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Convolution in Continuous and Discrete Time

Discrete-Time Convolution

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Continuous-Time Convolution

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Common Structure

- Multiply a shifted version of one signal
- Sum (DT) or integrate (CT)
- Output depends on overlap between signals

Equivalent Forms of Convolution

Discrete-Time

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Change of variable: let $m = n - k$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

Continuous-Time

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Change of variable: let $\lambda = t - \tau$

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

How to Compute Convolution (Mechanics)

To compute

$$y(t) = \int x(\tau)\, h(t-\tau)\, d\tau$$

We follow the steps:

1. **Flip** one signal:
$$h(\tau) \rightarrow h(-\tau)$$
2. **Shift** it by t:
$$h(-\tau) \rightarrow h(t-\tau)$$
3. **Multiply** with $x(\tau)$
4. **Integrate** over τ

Geometric View

Convolution measures the *overlapping area* between $x(\tau)$ and a shifted, flipped h.

DT Convolution Mechanics (Flip–Shift–Multiply–Sum)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

To compute $y[n]$:

1. Flip $h[k] \rightarrow h[-k]$
2. Shift by n: $h[-k] \rightarrow h[n-k]$
3. Multiply with $x[k]$
4. Sum over all k

Interpretation: $y[n]$ is a sliding weighted sum of $x[k]$.

Length of the Convolution (DT Case)

Let

$$x[n] \neq 0 \quad \text{for } n = n_x^{\text{min}}, \dots, n_x^{\text{max}}$$

$$h[n] \neq 0 \quad \text{for } n = n_h^{\text{min}}, \dots, n_h^{\text{max}}$$

Define lengths:

$$L_x = n_x^{\text{max}} - n_x^{\text{min}} + 1, \qquad L_h = n_h^{\text{max}} - n_h^{\text{min}} + 1$$

Non-zero index range:

$$y[n] \neq 0 \quad \text{for } n = n_y^{\text{min}}, \dots, n_y^{\text{max}}$$

with

$$n_y^{\text{min}} = n_x^{\text{min}} + n_h^{\text{min}}, \qquad n_y^{\text{max}} = n_x^{\text{max}} + n_h^{\text{max}}.$$

Result

The convolution $y[n] = (x * h)[n]$ is nonzero over

$$L_y = L_x + L_h - 1.$$

Convolution Examples (Practice)

Example 1 (Discrete-Time)
Compute the convolution $y[n] = (x * h)[n]$ for the finite-length sequences:

$$x[n] = \{1,\ 2,\ 1\}, \qquad h[n] = \{1,\ 1\}$$

- Assume both sequences start at $n = 0$ (i.e., $x[0] = 1$, $h[0] = 1$).

Example 2 (Continuous-Time)
Compute the convolution $y(t) = (x * h)(t)$ for:

$$x(t) = e^{-at}\, u(t), \qquad h(t) = u(t), \qquad a > 0$$

When Does Convolution Not Make Sense?

Discrete-Time Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

This sum must **converge** for every n. Convolution may **not exist** if:

- The series diverges (e.g. it is not absolutely summable)
- The terms do not decay sufficiently

Continuous-Time Convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The integral must **converge**. Convolution may fail if:

- The integral diverges
- Signals are not absolutely integrable

Discrete-Time Example

$$x[n] = 1, \quad h[n] = 1$$

$$y[n] = \sum_{k=-\infty}^{\infty} 1 \quad \rightarrow \quad \text{diverges}$$

Continuous-Time Example

$$x(t) = 1, \quad h(t) = 1$$

$$y(t) = \int_{-\infty}^{\infty} 1 \, d\tau \quad \rightarrow \quad \text{diverges}$$

Discrete-Time Example

$$x[n] = 1, \quad h[n] = 1$$

$$y[n] = \sum_{k=-\infty}^{\infty} 1 \quad \rightarrow \quad \text{diverges}$$

Continuous-Time Example

$$x(t) = 1, \quad h(t) = 1$$

$$y(t) = \int_{-\infty}^{\infty} 1 \, d\tau \quad \rightarrow \quad \text{diverges}$$

Convolution does not even make sense in those cases!

Properties of Convolution

Continuous time (CT):

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Discrete time (DT):

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Commutativity
$(x * h)(t) = (h * x)(t)$

Associativity
$((x * h) * g)(t) = (x * (h * g))(t)$

Distributivity
$(x*(h+g))(t) = (x*h)(t)+(x*g)(t)$

Identity element
$(x * \delta)(t) = x(t)$

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Identity element
$(x * \delta)[n] = x[n]$

Differentiation (CT) / Difference (DT) Property

Continuous Time (CT)

$$\frac{d}{dt}(x*h) = \left(\frac{dx}{dt}\right)*h = x*\left(\frac{dh}{dt}\right)$$

Discrete Time (DT)

Let $\Delta x[n] = x[n] - x[n-1]$.

$$\Delta(x*h) = (\Delta x)*h = x*(\Delta h)$$

Interpretation:

- CT: Differentiation distributes over convolution.
- DT: The first-difference operator distributes over convolution.

Time-Invariance Property of Convolution

Assume

$$z(t) = (x * y)(t) \qquad \text{and} \qquad z[n] = (x * y)[n].$$

Continuous Time (CT)

$$\begin{aligned} z(t - t_0) &= (x(\cdot - t_0) * y)(t) \\ &= (x * y(\cdot - t_0))(t) \end{aligned}$$

Discrete Time (DT)

$$\begin{aligned} z[n - n_0] &= (x[\cdot - n_0] * y)[n] \\ &= (x * y[\cdot - n_0])[n] \end{aligned}$$

Shifting either signal shifts the convolution result by the same amount.

Associativity of Convolution (Discrete-Time)

Claim

$$x[n] * (h[n] * g[n]) = (x[n] * h[n]) * g[n]$$

Start from the left-hand side

$$(h * g)[k] = \sum_{m=-\infty}^{\infty} h[m]\, g[k-m]$$

$$(x * (h * g))[n] = \sum_{k=-\infty}^{\infty} x[k]\, (h * g)[n-k]$$

Substitute:

$$= \sum_{k} x[k] \left(\sum_{m} h[m]\, g[(n-k) - m] \right)$$

Reorder the summations

$$= \sum_{m} h[m] \left(\sum_{k} x[k]\, g[n - (k+m)] \right)$$

Let $\ell = k + m$

$$= \sum_{m} h[m] \left(\sum_{\ell} x[\ell - m]\, g[n - \ell] \right)$$

$$= \sum_{\ell} (x * h)[\ell]\, g[n-\ell] = ((x * h) * g)[n]$$

Correlation: Measuring Similarity

Cross-Correlation (Discrete-Time)

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n+m]$$

Cross-Correlation (Continuous-Time)

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t+\tau)\, dt$$

Interpretation

- Slide one signal relative to the other
- Multiply (with complex conjugation)
- Sum (DT) or integrate (CT)

$$\text{Large value} \;\Rightarrow\; \text{strong similarity at that shift}$$

For real-valued signals, $y^* = y$. *Correlation is essentially an inner product between a signal and a shifted version of another signal.*

Example: Discrete-Time Cross-Correlation

Let the finite-length sequences be

$$x[n] = \{1,\ 2,\ 1\}, \qquad y[n] = \{0,\ 1,\ 2,\ 1\}$$

Assume both start at $n = 0$.
Compute the cross-correlation

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n+m]$$

Let's:

- Determine the valid range of m
- Compute $R_{xy}[m]$
- Identify the shift where correlation is maximum
- Compare with $R_{yx}[m]$

Length of the Cross-Correlation (DT Case)

Let

$$x[n] \neq 0 \quad \text{for } n = n_x^{\text{min}}, \ldots, n_x^{\text{max}}$$

$$y[n] \neq 0 \quad \text{for } n = n_y^{\text{min}}, \ldots, n_y^{\text{max}}$$

Define lengths:

$$L_x = n_x^{\text{max}} - n_x^{\text{min}} + 1$$

$$L_y = n_y^{\text{max}} - n_y^{\text{min}} + 1$$

Result The cross-correlation $R_{xy}[m]$ is nonzero for

$$m = n_y^{\text{min}} - n_x^{\text{max}}, \ldots, n_y^{\text{max}} - n_x^{\text{min}}$$

Therefore,

$$L_R = L_x + L_y - 1$$

Autocorrelation

Discrete-Time

$$R_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n+m]$$

Continuous-Time

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t+\tau)\, dt$$

Properties of Correlation

1. Symmetry Relations
Autocorrelation (Hermitian Symmetry)

$$R_{xx}[m] = R_{xx}^*[-m]$$

$$R_{xx}(\tau) = R_{xx}^*(-\tau)$$

If x is real-valued, this reduces to an even function.
Cross-Correlation Symmetry

$$R_{xy}[m] = R_{yx}^*[-m]$$

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

2. Auto-Correlation Value at Zero Shift (Energy)

$$R_{xx}[0] = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

3. Linearity Structure

$$R_{ax_1+bx_2,\ y} = aR_{x_1y} + bR_{x_2y}$$

$$R_{x,\ ay_1+by_2} = a^* R_{xy_1} + b^* R_{xy_2}$$

Linearity in first argument, conjugate-linearity in second.

Connection Between Correlation and Convolution

Discrete-Time

$$R_{xy}[m] = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n+m]$$

Rewrite $y[n+m] = y[-(m-n)]$:

$$R_{xy}[m] = \sum_{n} x[n]\, y^*[-(m-n)]$$

Define the time-reversed conjugate signal:

$$\tilde{y}[n] = y^*[-n]$$

Then,

$$R_{xy}[m] = (x * \tilde{y})[m]$$

Continuous-Time

$$R_{xy}(\tau) = \int x(t)\, y^*(t+\tau)\, dt = (x * \tilde{y})(\tau), \qquad \tilde{y}(t) = y^*(-t)$$

Correlation as an Inner Product / Projection

Discrete-Time Inner Product

$$\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$$

Correlation as an Inner Product / Projection

Discrete-Time Inner Product

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$$

Cross-Correlation

$$R_{xy}[m] = \sum_{n} x[n]\, y^*[n+m] = \langle x, y_m\rangle$$

where

$$y_m[n] = y[n+m]$$

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Key Interpretation

Correlation is the **inner product** between a signal and a shifted version of another signal.

Correlation as an Inner Product / Projection (2)

Cross-Correlation

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$$R_{xy}[m] = \sum_n x[n]\, y^*[n+m] = \langle x, y_m \rangle$$

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Key Interpretation

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Geometric View

$$R_{xy}[m] = \|x\|\,\|y\| \cos(\theta_m)$$

- Large magnitude $\Rightarrow$ strong alignment
- Zero $\Rightarrow$ orthogonality ($\cos(\theta_m) = 0$)
- Sign indicates direction (real signals)

Our first real problem: Detection Problem

We observe a noisy signal:

$$r(t) = s(t - \tau_0) + w(t)$$

- $s(t)$: known template
- τ_0: unknown delay
- $w(t)$: noise

Goal: Estimate the delay τ_0.

Key Idea

If we slide the template across $r(t)$ and measure similarity...

$$\text{Maximum similarity} \quad \Rightarrow \quad \text{correct delay}$$

Detection becomes a **similarity search problem**.

Similarity = Correlation

Measure similarity using correlation:

$$R_{rs}(t) = \int r(\tau)s^*(\tau - t)\, d\tau$$

- Slide $s(t)$ across $r(t)$
- Compute inner product at each shift
- Look for the peak

Matched Filter

Correlation can be implemented using an LTI system (filter):

$$h(t) = s^*(-t)$$

$$y(t) = (r * h)(t)$$

Matched Filter

Correlation can be implemented using an LTI system (filter):

$$h(t) = s^*(-t)$$

$$y(t) = (r * h)(t)$$

$$y(t) = \text{correlation between } r \text{ and } s$$

Geometric Interpretation

In discrete time:

$$y[n] = \sum_k r[k]\, s[k-n]$$

This is an inner product:

$$y[n] = \langle r, s_n \rangle$$

Maximum output ⇔ maximum alignment

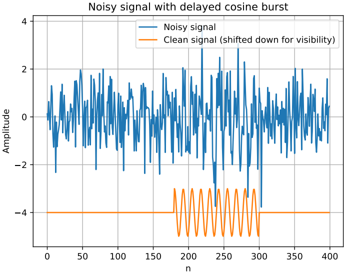
Signal vs Noise

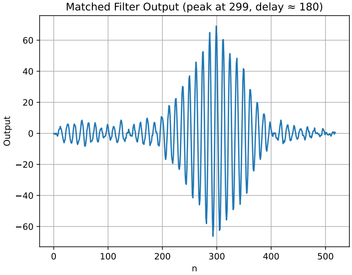
At the correct delay:

$$y(\tau_0) = \underbrace{\|s\|^2}_{\text{coherent sum}} + \underbrace{\text{noise}}_{\text{incoherent sum}}$$

- Signal adds coherently
- Noise averages out

Matched Filter (Example: Plots)





Thank you

- **Any Questions?**
- **Office Hours**:
  - **Mon & Tue** *(09:00-11:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_SS_AM*)
- **Material and Announcements**

Laboratory of Automation & Robotics