Signal & Systems

Lecture 2: Introduction to Systems

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What is a System?

Definition

A **system** is a rule that maps an input signal to an output signal:

$$x(t) \longrightarrow y(t) \qquad \text{or} \qquad x[n] \longrightarrow y[n]$$

$$y = \mathcal{T}\{x\}$$

Interpretation:

- A system is a *signal transformation*.
- It processes an input signal and produces an output signal.
- Mathematically, it is an *operator* acting on signals.

Examples:

- Amplifier: $y(t) = 3x(t)$
- Echo system: $y(t) = x(t) + 0.5\,x(t-1)$
- Moving average (DT): $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$

Systems in Cascade (Series Connection)

Definition

Two systems are connected in **cascade** (series) when the output of the first is the input of the second.

$$x \longrightarrow \mathcal{T}_1 \longrightarrow v \longrightarrow \mathcal{T}_2 \longrightarrow y$$

$$v = \mathcal{T}_1\{x\} \qquad y = \mathcal{T}_2\{v\}$$

$$\boxed{y = \mathcal{T}_2\{\mathcal{T}_1\{x\}\}}$$

Interpretation:

- The signal is processed step-by-step.
- Order generally matters.

Systems in Parallel

Definition

Two systems are connected in **parallel** when the same input is processed by both systems and the outputs are added (or subtracted).

$$x$$

$$\downarrow$$

$$\begin{cases} \mathcal{T}_1 \\ \mathcal{T}_2 \end{cases} \quad \rightarrow \quad y = y_1 + y_2$$

$$y_1 = \mathcal{T}_1\{x\} \qquad y_2 = \mathcal{T}_2\{x\}$$

$$\boxed{y = \mathcal{T}_1\{x\} + \mathcal{T}_2\{x\}}$$

Interpretation:

- The signal is processed in different ways.
- Outputs are combined.

Feedback Connection

Definition

In a **feedback system**, the output is processed by a feedback block and combined with the input at a summing junction (parallel).

$$\text{Error (input to forward system):} \quad u = x - v$$

$$\text{Forward system:} \quad y = \mathcal{T}_1\{u\}$$

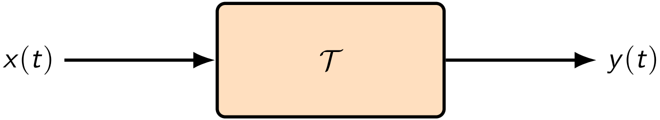
$$\text{Feedback signal:} \quad v = \mathcal{T}_2\{y\}$$

$$\boxed{y = \mathcal{T}_1\{x - \mathcal{T}_2\{y\}\}}$$

Interpretation:

- The system output influences future behavior.
- Feedback can amplify or stabilize a system.
- Very common in control systems.

System interconnections can be visualized as block diagrams:



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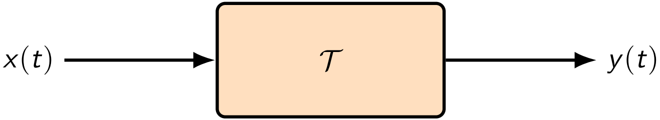
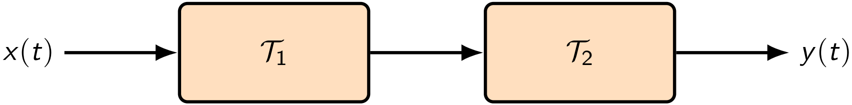
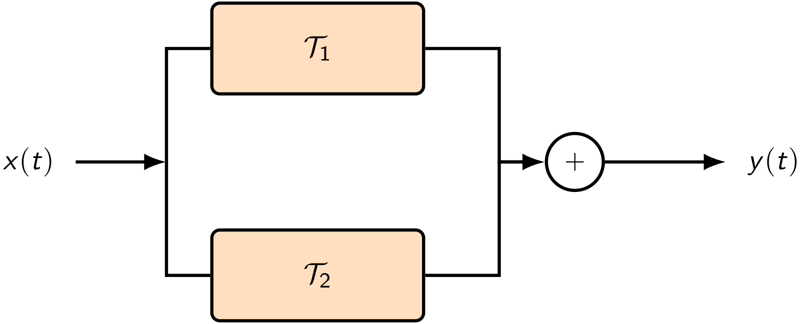


Cascade:



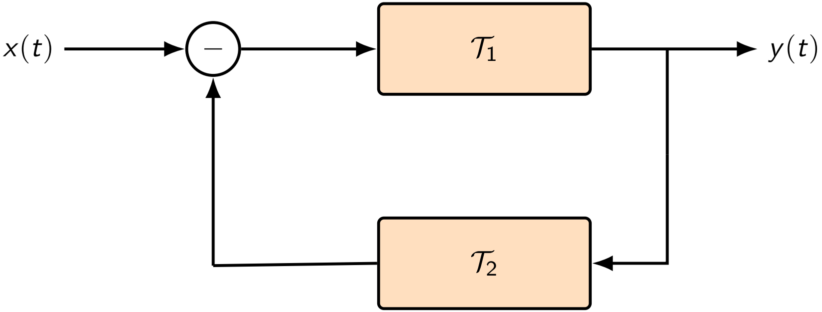
System Graphical Examples (2)

Parallel:



System Graphical Examples (3)

Feedback:



Why Study System Interconnections?

- Real systems are rarely simple.
- Complex systems are built from simpler blocks.
- Understanding interconnections allows us to:
  - Simplify systems
  - Analyze behavior
  - Predict stability and response

Key Idea

If we understand simple systems well, we can understand complex systems.

Taxonomy of Systems (Overview)

A **system** maps an input signal to an output signal:

$$y = \mathcal{T}\{x\}$$

We often classify systems based on **how they behave** and **what assumptions we can make**.

Common classification axes

- Deterministic vs. stochastic
- Single-/multi-input and single-/multi-output (SISO/MIMO)
- Static (memoryless) vs. dynamic (with memory)
- Causal vs. anti-causal vs. non-causal
- Linear vs. non-linear
- Time-invariant vs. time-varying

Deterministic vs. Stochastic Systems

Deterministic

For a given input, the output is **fully determined**.

$$y = \mathcal{T}\{x\} \quad \text{(no randomness)}$$

Stochastic (Random)

The output is **not fully predictable** even for a fixed input:

$$y = \mathcal{T}\{x, \omega\}$$

where ω represents randomness (noise, random parameters, etc.).

Examples:

- Deterministic: $y[n] = 3x[n]$
- Stochastic: $y[n] = x[n] + w[n]$ (noise $w[n]$)

SISO vs. MIMO Systems

SISO (Single-Input Single-Output)

One input signal → one output signal.

$$x \to y$$

MIMO (Multiple-Input Multiple-Output)

Multiple signals in, multiple signals out:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \longrightarrow \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_P \end{bmatrix}$$

Examples:

- SISO: audio filter (one waveform in, one waveform out)
- MIMO: stereo audio (2-in/2-out), multi-antenna systems

Static (Memoryless) vs. Dynamic (With Memory)

Static (Memoryless)

The output at time t (or index n) depends only on the input at the **same** time:

$$y(t) = f(x(t)), \qquad y[n] = f(x[n])$$

Dynamic (With Memory)

The output depends on **past/future** input/output values:

$$y(t) \text{ depends on } x(\tau) \text{ or } y(\tau) \text{ for } \tau \neq t$$

$$y[n] \text{ depends on } x[k] \text{ or } y[k] \text{ for } k \neq n$$

Examples:

- Memoryless: $y[n] = (x[n])^2$
- With memory: $y[n] = x[n] + 0.5x[n-1]$

Causal, Anti-Causal, Non-Causal Systems

Causal

Output at time t depends only on **present and past** inputs and/or outputs:

$$y(t_0) \text{ depends only on } x(t) \text{ or } y(t),\ \ t \leqslant t_0$$

$$y[n_0] \text{ depends only on } x[n] \text{ or } y[n],\ \ n \leqslant n_0$$

Anti-causal / Non-causal

- **Anti-causal:** depends only on future inputs/outputs
- **Non-causal:** depends on past and future inputs/outputs

Examples:

- Causal: $y[n] = x[n] + x[n-1]$
- Anti-causal: $y[n] = x[n+1]$
- Non-causal: $y[n] = x[n+1] + x[n-1]$

Linear vs. Non-linear Systems

Linearity (Superposition)

A system is **linear** if for any signals x_i and scalars a_i:

$$\mathcal{T}\{\sum_i a_i x_i\} = \sum_i a_i \mathcal{T}\{x_i\}$$

Examples:

- Linear: $y[n] = 2x[n] - x[n-1]$
- Non-linear: $y[n] = (x[n])^2$
- Non-linear: $y[n] = |x[n]|$

Tip: Check linearity with the *superposition test*.

Time-Invariant vs. Time-Varying Systems

Time Invariance

If shifting the input by t_0 shifts the output by the **same amount**:

$$x(t) \rightarrow y(t) \quad \Rightarrow \quad x(t-t_0) \rightarrow y(t-t_0)$$

(and similarly for DT: $x[n-n_0] \rightarrow y[n-n_0]$)

Time-Varying (Time-Dependent)

If the above property does **not** hold, the system is time-varying.

Examples:

- Time-invariant: $y(t) = x(t-1)$
- Time-varying: $y(t) = t\,x(t)$

Quick Practice: Classify These Systems

Classify each system as: memoryless/dynamic, causal/non-causal, linear/non-linear, time-invariant/time-varying.

1. $y[n] = 3x[n]$
2. $y[n] = x[n] + x[n-1]$
3. $y[n] = x[n+1]$
4. $y(t) = t\,x(t)$
5. $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

Systems Defined by Differential Equations

Continuous-Time Systems

Many physical systems are described by **differential equations** relating input and output:

$$a_N \frac{d^N y(t)}{dt^N} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + \cdots + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}$$

Interpretation:

- Output depends on input and its derivatives.
- Output may also depend on its own derivatives.
- The highest derivative of $y(t)$ determines the **order**.

Example (RC circuit):

$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

Discrete-Time Systems

Discrete-time systems are often described by **difference equations**:

$$a_N y[n-N] + \cdots + a_1 y[n-1] + a_0 y[n] = b_M x[n-M] + \cdots + b_0 x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]$$

Interpretation:

- Output depends on current and past inputs.
- Output may depend on past outputs.
- The largest delay determines the **order**.

Example (Simple filter):

$$y[n] = x[n] + 0.5\, y[n-1]$$

Stability

Bounded Signal

A signal is **bounded** if there exists $M < \infty$ such that:

$$|x(t)| \leqslant M \quad \forall t \qquad \text{or} \qquad |x[n]| \leqslant M \quad \forall n$$

BIBO Stability

A system is **Bounded-Input Bounded-Output (BIBO) stable** if:

$$\text{bounded input} \Rightarrow \text{bounded output}$$

Interpretation:

- A finite input should not produce an infinite output.
- Physically: the system does not "blow up."

Examples of Stability

Example 1 (Stable):

$$y[n] = 0.5\,x[n]$$

Bounded input $\Rightarrow$ bounded output.

Example 2 (Unstable):

$$y[n] = n\,x[n]$$

If $x[n] = 1$, then $y[n] = n$ (unbounded).

Example 3 (Dynamic Stable System):

$$y[n] = x[n] + 0.5\,y[n-1]$$

Output decays over time $\Rightarrow$ stable.

Example 4 (Unstable Recursion):

$$y[n] = x[n] + 1.5\,y[n-1]$$

Output grows exponentially $\Rightarrow$ unstable.

Physical Intuition Behind Stability

- Stable systems dissipate energy.
- Unstable systems amplify energy without bound.

Engineering Perspective

Stability is essential:

- Control systems
- Communication systems
- Filters

Definition

A system is **LTI** if it satisfies:

1. **Linearity** (Superposition)
2. **Time Invariance**

Linearity:

$$\mathcal{T}\{ax_1 + bx_2\} = a\mathcal{T}\{x_1\} + b\mathcal{T}\{x_2\}$$

Time Invariance:

$$x(t) \to y(t) \quad \Rightarrow \quad x(t - t_0) \to y(t - t_0)$$

Intuition Behind LTI Systems

Linearity:

- Scaling input ⇒ scaled output
- Adding inputs ⇒ outputs add

Time invariance:

- System behavior does not depend on when the signal is applied.
- The system has no explicit dependence on time.

Engineering Interpretation

LTI systems are predictable, composable, and analyzable.

Total Response of a System

For linear systems described by differential or difference equations:

Total response = **Zero-input response** + **Zero-state response**

Zero-Input Response

Response due only to **initial conditions** (input is zero).

Zero-State Response

Response due only to the **input** (initial conditions are zero).

Zero-Input (Natural) Response

Set the input to zero:

$$x(t) = 0 \quad \text{or} \quad x[n] = 0$$

Solve the homogeneous equation.

Example (RC circuit):

$$RC\,\frac{dy(t)}{dt} + y(t) = 0$$

Solution:

$$y(t) = Ae^{-t/RC}$$

Interpretation

This response is determined by stored energy (initial voltage in the capacitor).

Zero-State (Forced) Response

Set initial conditions to zero.

Solve the equation using only the input.
Example (DT first-order system):

$$y[n] = x[n] + 0.5\, y[n-1]$$

If $y[-1] = 0$, the response depends only on $x[n]$.

Interpretation

This is the response caused by the input signal.

Impulse Response

Definition

The **impulse response** $h(t)$ (or $h[n]$) is the output of a system when the input is an impulse:

$$x(t) = \delta(t) \quad \Rightarrow \quad y(t) = h(t)$$

Important:

- Initial conditions are assumed zero.
- Impulse response is a zero-state response.

Why Is the Impulse Response So Important?

For **LTI systems**:

Key Result

The impulse response **completely characterizes the system**.

If we know $h(t)$ or $h[n]$, we can compute the output for *any* input.

Deriving Convolution from LTI Systems

We have a signal $x(t)$ and an LTI system $y(t) = \mathcal{T}\{x(t)\}$.

Step 1: Decompose any signal using impulses

Any (well-behaved) continuous-time signal can be written as a weighted superposition of shifted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Step 2: Define the impulse response

For an LTI system, the **impulse response** is the output to $\delta(t)$ (zero-state):

$$\delta(t) \;\longrightarrow\; h(t) = \mathcal{T}\{\delta(t)\}$$

Deriving Convolution from LTI Systems (2)

Step 3: Use time invariance (shifting property)
If $\delta(t) \rightarrow h(t)$, then shifting the input by τ shifts the output by the same amount:

$$\delta(t-\tau) \longrightarrow h(t-\tau)$$

Step 4: Use linearity (superposition)
Using $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$, we get:

$$\begin{aligned} y(t) &= \mathcal{T}\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\} \\ &= \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{\text{No dependence on } t} \mathcal{T}\{\delta(t-\tau)\}\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \end{aligned}$$

Convolution Integral

$$\boxed{y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

How to Compute Convolution (Mechanics)

To compute

$$y(t) = \int x(\tau)\, h(t-\tau)\, d\tau$$

We follow the steps:

1. **Flip** one signal:

$$h(\tau) \rightarrow h(-\tau)$$

2. **Shift** it by t:

$$h(-\tau) \rightarrow h(t-\tau)$$

3. **Multiply** with $x(\tau)$
4. **Integrate** over τ

Geometric View

Convolution measures the *overlapping area* between $x(\tau)$ and a shifted, flipped h.

Example: Rectangular Pulse * Rectangular Pulse

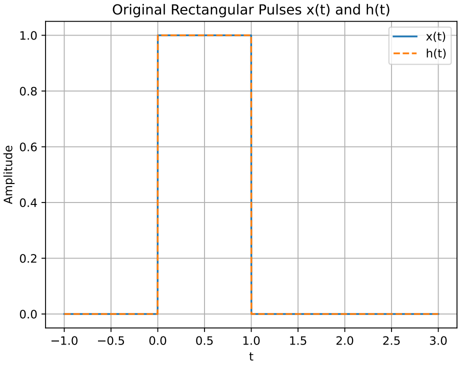
Let

$$x(t) = \begin{cases} 1, & 0 \leqslant t \leqslant T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1, & 0 \leqslant t \leqslant T \\ 0, & \text{otherwise} \end{cases}$$

Compute:

$$y(t) = x(t) * h(t)$$



Step 1-2: Flip and Shift

Flip:

$$h(\tau) \rightarrow h(-\tau)$$

Now shift by t:

$$h(t-\tau)$$

We now compute the overlap between:

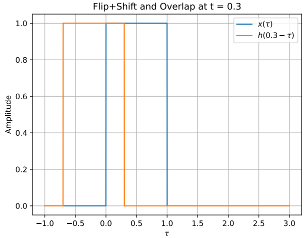
$$x(\tau) \quad \text{and} \quad h(t-\tau)$$

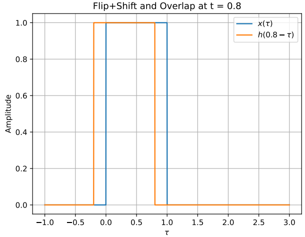
The value of the convolution at time t equals the **area of overlap**.

Computing the Overlap

The overlap depends on t:

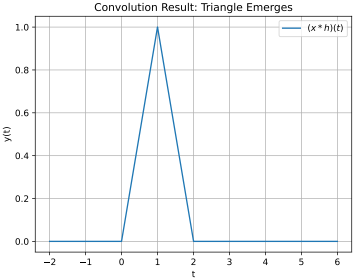
- For $t < 0$: no overlap $\Rightarrow y(t) = 0$
- For $0 \leqslant t \leqslant T$: overlap grows linearly
- For $T \leqslant t \leqslant 2T$: overlap decreases linearly
- For $t > 2T$: no overlap $\Rightarrow y(t) = 0$





Result: A Triangular Signal

$$y(t) = \begin{cases} t, & 0 \leqslant t \leqslant T \\ 2T - t, & T \leqslant t \leqslant 2T \\ 0, & \text{otherwise} \end{cases}$$



Important Observation

Convolution of two rectangular pulses produces a **triangle**.

Solving the Convolution: Rectangle * Rectangle

Let

$$x(t) = \begin{cases} 1, & 0 \leqslant t \leqslant T \\ 0, & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 1, & 0 \leqslant t \leqslant T \\ 0, & \text{otherwise} \end{cases}$$

We compute

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Since $x(\tau) = 1$ only for $\tau \in [0, T]$ and $h(t-\tau) = 1$ only when

$$0 \leqslant t - \tau \leqslant T \iff t - T \leqslant \tau \leqslant t,$$

the integrand equals 1 only when both conditions hold:

$$\tau \in [0, T] \cap [t - T,\ t].$$

Therefore,

$$y(t) = \text{length}\big([0, T] \cap [t - T,\ t]\big).$$

Solving the Convolution: Piecewise Result

Compute the overlap length in different ranges of t:

- $t < 0$: no overlap $\Rightarrow y(t) = 0$
- $0 \leqslant t \leqslant T$: overlap is $[0, t]$ (length t) $\Rightarrow y(t) = t$
- $T \leqslant t \leqslant 2T$: overlap is $[t - T, T]$ (length $2T - t$) $\Rightarrow y(t) = 2T - t$
- $t > 2T$: no overlap $\Rightarrow y(t) = 0$

Final Answer

$$y(t) = \begin{cases} 0, & t < 0, \\ t, & 0 \leqslant t \leqslant T, \\ 2T - t, & T \leqslant t \leqslant 2T, \\ 0, & t > 2T. \end{cases}$$

Deriving Convolution from LTI Systems (DT)

We have a signal $x[n]$ and an LTI system $y[n] = \mathcal{T}\{x[n]\}$.

Step 1: Decompose any sequence using impulses

Any discrete-time signal can be written as a weighted sum of shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Step 2: Impulse response

For an LTI system:

$$\delta[n] \longrightarrow h[n] = \mathcal{T}\{\delta[n]\}$$

Step 3: Time invariance

$$\delta[n-k] \longrightarrow h[n-k]$$

Step 4: Linearity

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,\mathcal{T}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Convolution Sum

$$\boxed{y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$$

DT Convolution Mechanics (Flip–Shift–Multiply–Sum)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

To compute $y[n]$:

1. Flip $h[k] \rightarrow h[-k]$
2. Shift by n: $h[-k] \rightarrow h[n-k]$
3. Multiply with $x[k]$
4. Sum over all k

Interpretation: $y[n]$ is a sliding weighted sum of $x[k]$.

Example: Discrete-Time Convolution (Step-by-Step)

Let the finite-length sequences be

$$x[n] = \{1,\ 2,\ 1\} \quad \text{for } n = 0, 1, 2$$

$$h[n] = \{1,\ 1\} \quad \text{for } n = 0, 1$$

Compute

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k].$$

When Is $y[n]$ Nonzero?

For $y[n]$ to be nonzero, there must exist at least one k such that:

$$x[k] \neq 0 \quad \text{and} \quad h[n-k] \neq 0.$$

We know:

$$x[k] \neq 0 \quad \text{only if } 0 \leqslant k \leqslant 2$$

$$h[n-k] \neq 0 \quad \text{only if } 0 \leqslant n-k \leqslant 1$$

The second condition implies:

$$0 \leqslant n-k \leqslant 1 \quad \iff \quad n-1 \leqslant k \leqslant n.$$

Therefore, for $y[n]$ to be nonzero:

$$k \in [0,2] \cap [n-1,n].$$

The output $y[n]$ is nonzero only if:

$$[0,2] \cap [n-1,n] \neq \varnothing.$$

This happens only when:

$$0 \leqslant n \leqslant 3.$$

Length of the Convolution (DT Case)

Let

$$x[n] \neq 0 \quad \text{for } n = n_x^{\min}, \ldots, n_x^{\max}$$

$$h[n] \neq 0 \quad \text{for } n = n_h^{\min}, \ldots, n_h^{\max}$$

Define lengths:

$$L_x = n_x^{\max} - n_x^{\min} + 1, \qquad L_h = n_h^{\max} - n_h^{\min} + 1$$

Non-zero index range:

$$y[n] \neq 0 \quad \text{for } n = n_y^{\min}, \ldots, n_y^{\max}$$

with

$$n_y^{\min} = n_x^{\min} + n_h^{\min}, \qquad n_y^{\max} = n_x^{\max} + n_h^{\max}.$$

Result

The convolution $y[n] = (x * h)[n]$ is nonzero over

$$L_y = L_x + L_h - 1.$$

Example: Discrete-Time Convolution (Compute $y[n]$)

Using $h[0] = 1,\ h[1] = 1$ (and $h[n] = 0$ otherwise):

$$y[n] = \sum_k x[k]\, h[n-k] = x[n]h[0] + x[n-1]h[1] = x[n] + x[n-1].$$

Now compute each output sample:

$$\begin{aligned} y[0] &= x[0] + x[-1] = 1 + 0 = 1 \\ y[1] &= x[1] + x[0] = 2 + 1 = 3 \\ y[2] &= x[2] + x[1] = 1 + 2 = 3 \\ y[3] &= x[3] + x[2] = 0 + 1 = 1 \end{aligned}$$

Result

$$y[n] = \{1, 3, 3, 1\}$$

Convolution: Two Equivalent Views

System-output view (LTI)

$$y = \mathcal{T}\{x\} \quad \Longleftrightarrow \quad y = x * h$$

Interpretation:

- $x(\tau)$ tells us *how much* impulse occurs at time τ.
- $h(t-\tau)$ is the response of the system to an impulse at τ.
- The output is the sum/integral of all these shifted responses.

Properties of Convolution

Continuous time (CT):

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Discrete time (DT):

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Commutativity
$(x * h)(t) = (h * x)(t)$

Associativity
$((x * h) * g)(t) = (x * (h * g))(t)$

Distributivity
$(x*(h+g))(t) = (x*h)(t)+(x*g)(t)$

Identity element
$(x * \delta)(t) = x(t)$

Commutativity
$(x * h)[n] = (h * x)[n]$

Associativity
$((x * h) * g)[n] = (x * (h * g))[n]$

Distributivity
$(x*(h+g))[n] = (x*h)[n]+(x*g)[n]$

Identity element
$(x * \delta)[n] = x[n]$

Differentiation (CT) / Difference (DT) Property

Continuous Time (CT)

$$\frac{d}{dt}(x*h) = \left(\frac{dx}{dt}\right)*h = x*\left(\frac{dh}{dt}\right)$$

Discrete Time (DT)

Let $\Delta x[n] = x[n] - x[n-1]$.

$$\Delta(x*h) = (\Delta x)*h = x*(\Delta h)$$

Interpretation:

- CT: Differentiation distributes over convolution.
- DT: The first-difference operator distributes over convolution.

Time-Invariance Property of Convolution

Assume

$$z(t) = (x * y)(t) \qquad \text{and} \qquad z[n] = (x * y)[n].$$

Continuous Time (CT)

$$\begin{aligned} z(t - t_0) &= (x(\cdot - t_0) * y)(t) \\ &= (x * y(\cdot - t_0))(t) \end{aligned}$$

Discrete Time (DT)

$$\begin{aligned} z[n - n_0] &= (x[\cdot - n_0] * y)[n] \\ &= (x * y[\cdot - n_0])[n] \end{aligned}$$

Shifting either signal shifts the convolution result by the same amount.

Thank you

- **Any Questions?**
- **Office Hours**:
  - **Mon & Tue** *(09:00-11:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_SS_AM*)
- **Material and Announcements**

Laboratory of Automation & Robotics